1) $$[\nabla_1, \nabla_2] = C_1 \sqrt{EG-F^2} \cdot (90^\circ \text{ CW rotation})$$

$$= C_1 \begin{pmatrix} F & G \\ -E & -F \end{pmatrix}$$

2) $C_2 = \dfrac{\text{rotation}}{\text{unit area}}$ from parallel transport.

3) $K =$ det W

Last time showed $C_1 = C_2$

HW: $C_1 = K$.

Big thm: $C_1 = C_2 = K$.

Theorema Egregium

Formulas for K.

$$\nabla_1 = \partial_1 + \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 \end{pmatrix} = \partial_1 + A_1$$

$$\nabla_2 = \partial_2 + \begin{pmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 \end{pmatrix} = \partial_2 + A_2$$

$$[\nabla_1, \nabla_2] = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$$

$$= \begin{pmatrix} KF & KG \\ -KE & -KF \end{pmatrix}$$

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix} \quad \text{, etc.}$$

$$\begin{pmatrix} \Gamma_{111} \\ \Gamma_{112} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix}$$

$$\boxed{\Gamma_{ijk}} = \sum_{\ell} g_{k\ell} \Gamma_{ij}^{\ell} \qquad \text{, where } \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$= \boxed{\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)/2}$$

$$\Gamma_{111} = \left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right)/2 = E_u/2$$

$$\Gamma_{112} = \left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right)/2 = F_u - \frac{E_v}{2}$$

Conformal: $E = G$.

$$K = -\frac{1}{2G}(\ln G)_{uu+vv}$$

Case 1: Pseudosphere. $ds^2 = \dfrac{du^2 + dv^2}{v^2}$

$$E = G = \frac{1}{v^2}$$

$$\ln G = -2\ln v \qquad (\ln G)_u = 0$$

$$(\ln G)_v = \frac{-2}{v}$$

$$(\ln G)_{vv} = \frac{2}{v^2} \qquad K = -\frac{1}{2\left(\frac{1}{v^2}\right)} \cdot \frac{2}{v^2} = -1$$

Case 2: $$ds^2 = \frac{4}{(1+u^2+v^2)^2}\left(du^2+dv^2\right)$$

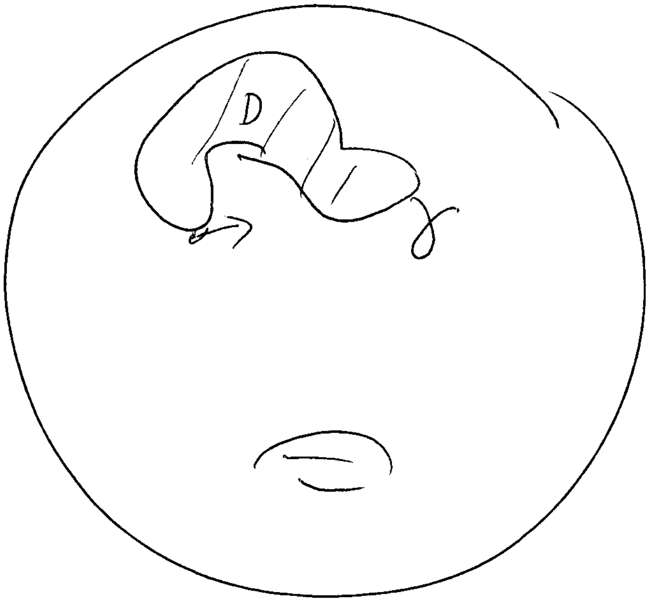
$$\ln G = \ln 4 - 2\ln(1+u^2+v^2)$$

$$(\ln G)_u = \frac{-4u}{(1+u^2+v^2)} \qquad (\ln G)_{uu} = \frac{-4\left[1+u^2+v^2-2u^2\right]}{(1+u^2+v^2)^2}$$

$$= -\frac{4(1+v^2-u^2)}{(1+u^2+v^2)^2}$$

$$(\ln G)_{vv} = \frac{-4(1+u^2-v^2)}{(\quad)^2}$$

$$K = -\frac{1}{2G}\,\frac{-8}{(\quad)^2} = 1$$



<u>local Gauss-Bonnet Thm:</u>

$$\int_\gamma K_g \, ds = 2\pi - \int_D K \, d\text{Area}$$

<u>Pf:</u> $K_g = \cancel{\text{rot}} \left(\nabla_\gamma \dot\gamma \cdot \vec{n} \right)$

$=$ rate at which $\dot\gamma$ rotates w.r.t. a covariantly constant reference $W(t)$

$= d\theta/dt$, where $\theta =$ angle between $\dot\gamma$ and reference vector.

$$\cos\theta = \langle \dot\gamma, w \rangle$$

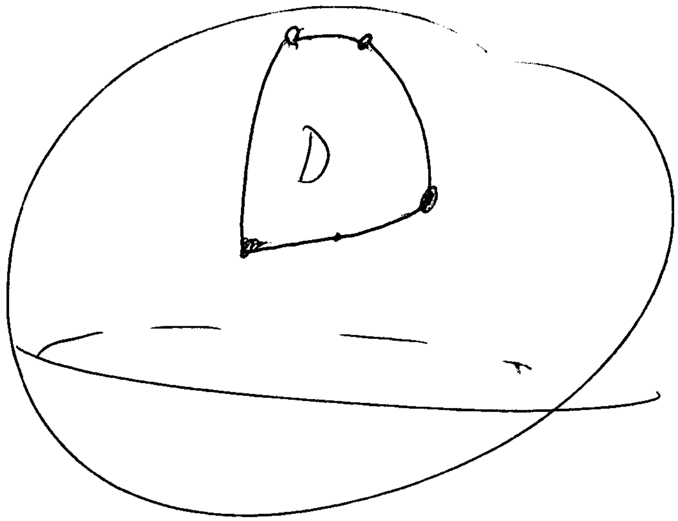
$$\frac{d}{dt}\cos\theta = \langle \nabla_\gamma \dot\gamma, w \rangle + \langle \dot\gamma, \cancel{\nabla_\gamma w} \rangle \to 0$$

$$-\sin\theta\,\dot\theta \quad = -K_g \sin\theta$$

$$\dot\theta = K_g$$

$$\theta(T) - \theta(0) = \int_\gamma K_g\,ds = \text{rotation of } \dot\gamma - \text{rotation of } w$$

$$= 2\pi - \int K \, d\text{area}$$



Thm for a geodesic polygon

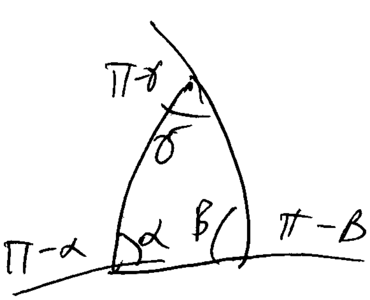
$$\sum \text{exterior angles} = 2\pi - \int_D K \, d\,\text{area}$$

Pf $\lim \int_\gamma K_g \, ds = \sum$ exterior angles

for smooth γ approximating polygon.

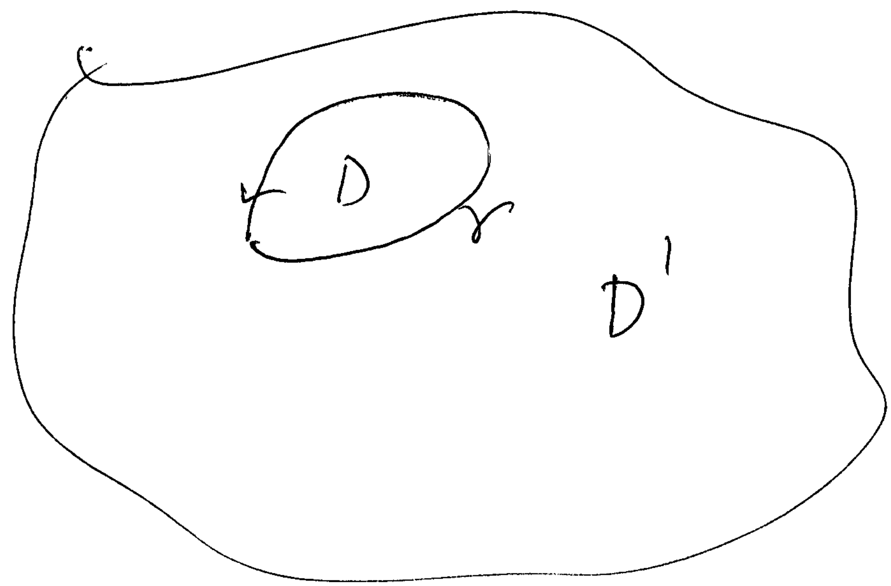
For a triangle, $\sum$ interior angles

$$= \pi + \int_D K \, d\,\text{area}$$



$$3\pi - (\alpha+\beta+\gamma) = 2\pi - \int K \, d\,\text{area}$$

$$\alpha+\beta+\gamma = \pi + \int K \, d\,\text{area}$$



$$\int_\gamma K_g\,ds = 2\pi - \int_D K\,d\,\text{area}$$

$$= -\left(2\pi - \int_{D'} K\,d\,\text{area}\right)$$

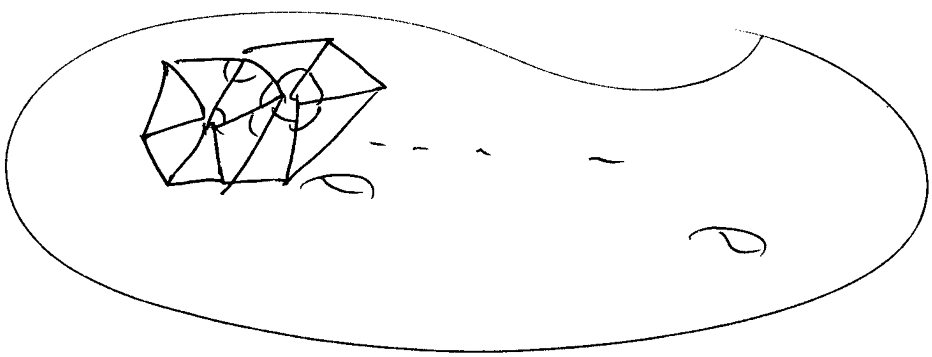
$$\boxed{\int_S K\,d\text{area} = 4\pi}$$

Global Gauss - Bonnet.

Thm For any compact surface,

$$\int K \, d(\text{area}) = 2\pi \chi(S),$$

where $\chi(S)$ = Euler characteristic
$= (\#\text{vertices} - \#\text{edges} + \#\text{faces})$
$\quad\quad V \quad\quad E \quad\quad F$



Triangulate surface.

$2E = 3F$

$F = 2E - 2F$

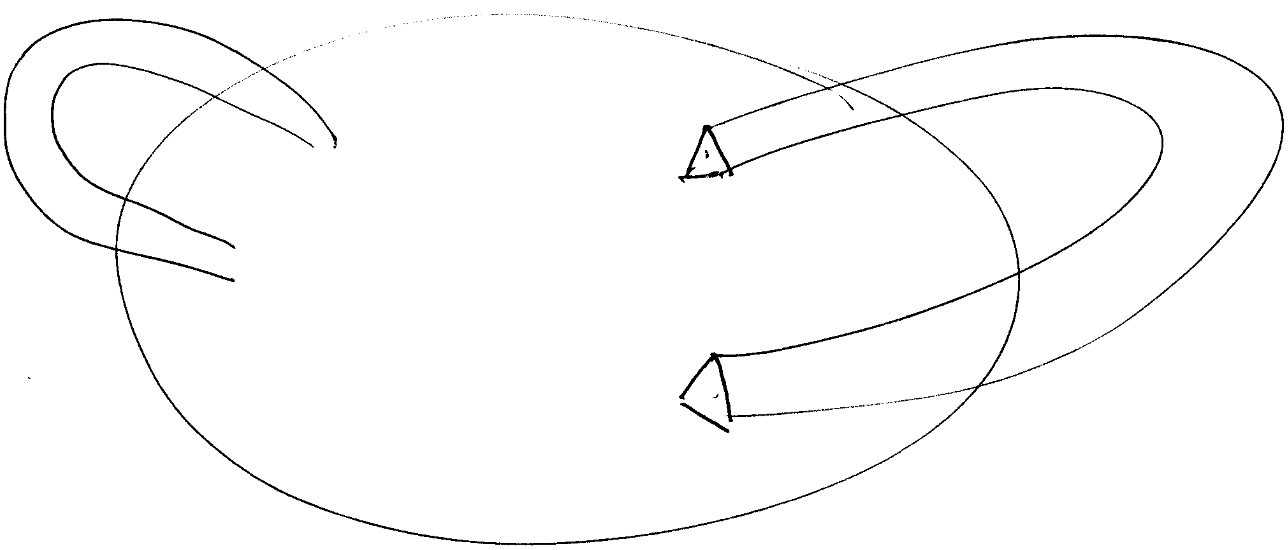
$$\sum \text{all angles} = \sum_{\text{vertices}} 2\pi$$

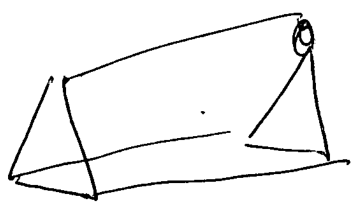
$$= \sum_{\text{triangles}} \left(\pi + \int K\right)$$

$$2\pi V = \pi F + \int_S K$$

$$= 2\pi E - 2\pi F + \int_S K$$

$$\int_S K = 2\pi(V - E + F) \checkmark$$





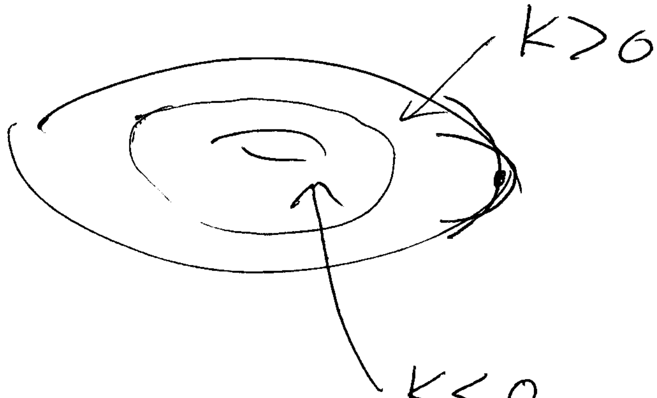
To add handle,
cut out 2 triangles
add 3 faces
add 3 edges

$V \to V$

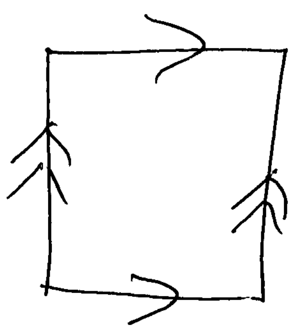
$E \to E+3$ $\qquad \chi \to \chi - 2$

$F \to F+1$

$$\chi(\text{g-holed torus}) = 2 - 2g = \begin{cases} >0 & g=0 \\ =0 & g=1 \\ <0 & g>1 \end{cases}$$



$$\int K \cancel{\partial} = 0$$



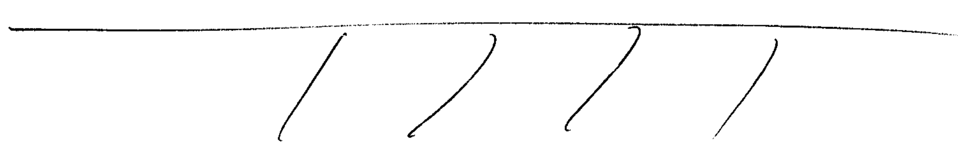
$ds^2 = dx^2 + dy^2 \qquad K=0$

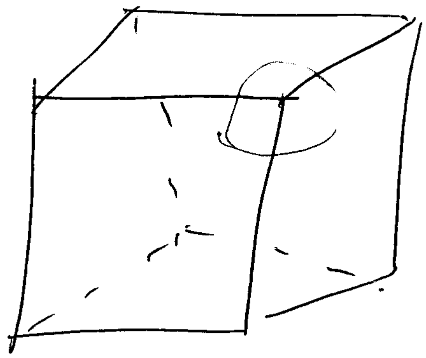
$\mathbb{R}^2/\mathbb{Z}^2$

$g>1$ want $K=-1$

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

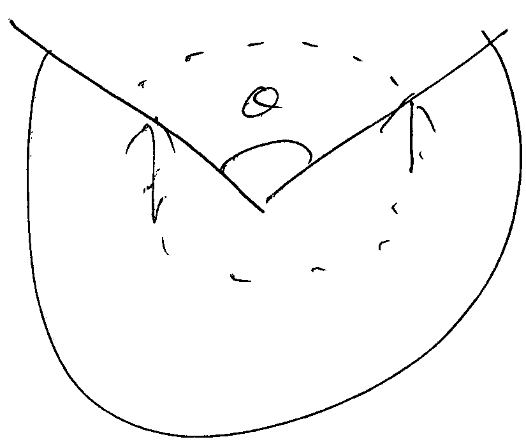
$T_g = \mathbb{H}^2/(\text{group})$





Curvature concentrates on corners.

Curvature of corner = angle defect. = θ



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m n-gons at a vertex.

n \ m	3	4	5	6
3	180°	120	60	0
4	90	0	−90	
5	~~5~~36	<0	<0	<0
6	0	<0	<0	<0