M365G Homework, due May 1, 2012

1. Let S_1 be a surface and p be a point on that surface. Show that there is a direct isometry of $\mathbb{R}^3$ that sends p to the origin and that sends a neighborhood of p in S_1 to a surface S_2 of the form $z = f(x, y)$, where $f(x, y) = ax^2/2 + by^2/2 + O(r^3)$, with $O(r^3)$ meaning terms that go to zero at least as fast as $(x^2 + y^2)^{3/2}$ as $x, y \to 0$. More precisely, it means that $|f(x, y) - (ax^2/2 + by^2/2)|/(x^2 + y^2)^{3/2}$ is bounded in a neighborhood of the origin. [Note: since everything is smooth, there is a Taylor series for $f(x, y)$. The expression $O(r^3)$ describes all the terms that go as $x^i y^j$ with $i + j \geq 3$. This also means that the derivatives of the $O(r^3)$ terms are $O(r^2)$, and that the second derivatives are $O(r)$.]

In the rest of this problem set, your answers should all be of the form (Some quantity) = (Some expression involving a, b, x, y) + $O(r^{\text{some power}})$. Don't forget that $(1 + \epsilon)^n = 1 + n\epsilon + O(\epsilon^2)$. This is particularly useful for $n = 1/2$ and $n = -1$.

2. Using coordinates $u = x$ and $v = y$, find expressions for the first and second fundamental forms of S_2 as a function of x, y, and compute the Gauss curvature K.

3. Compute all the Christoffel symbols for S_2 (see Prop 7.4.4), and compute the commutator $[\nabla_1, \nabla_2]$. Your answer should be a 2×2 matrix, from which you can infer the value of C_1 (as defined in class).

4. Show that the Gauss equations (Prop. 10.1.2, not to be confused with the Gauss equations of Prop 7.4.4 – Gauss had a lot of equations!) apply to S_2 at the origin. An earlier version of this problem also asked about the Codazzi-Mainardi equation. Do NOT evaluate those, as the expressions depend strongly on the $O(r^3)$ terms in $f(x, y)$. Also, this is practically the same calculation as problem 3. Either problem is enough to conclude that $K = C_1$ at the origin.

5. Returning to the original surface S_1, show that $C_1(p) = K(p)$. Conclude that C_1 and K are the same geometric quantity for all points on all surfaces.

6. Geodesics on S_2 are approximated very well by intersections of S_1 with vertical planes through the origin. That is, the shortest path from $(0, 0, 0)$ to (x_0, y_0, z_0) has x/y constant. Taking this result for granted, compute the distance from the origin to $(x, y, f(x, y))$.

7. Using the results of Problem 6, construct geodesic normal coordinates around the origin of S_2.

8. Now consider the "circle" obtained by fixing a value of r in the geodesic normal coordinates. [Note that in this context r is the geodesic distance from the origin, which is NOT the same as $\sqrt{x^2+y^2}$. This isn't a repeat of an exam problem!] Show that the circumference of that "circle" is
$2\pi r(1 - abr^2/6)+$ higher order. This shows that the defect in the circumference is proportional to the Gauss curvature.

9. Finally, compute the area enclosed by the circle [hint: $\int$ (circumference) dr] and the isoperimetric ratio Area/(circumference)^2.

Your answer to problem 9 SHOULD match the results of the exam problem. In the exam, we considered the intersection of a cylinder $x^2 + y^2 = r^2$ with the surface. That resulted in a curve that approximates the "circle" of problems 8 and 9. That approximation wasn't good enough to compute the circumference and area individually, but WAS good enough to compute the isoperimetric ratio. This is because a circle maximizes the ratio, so deviations from that circular shape only affect the ratio to second order.