M365G Homework Solutions

*1. Let $S_1$ be a surface and $p$ be a point on that surface. Show that there is a direct isometry of $\mathbb{R}^3$ that sends $p$ to the origin and that sends a neighborhood of $p$ in $S_1$ to a surface $S_2$ of the form $z = f(x,y)$, where $f(x,y) = ax^2/2 + by^2/2 + O(r^3)$, with $O(r^3)$ meaning terms that go to zero at least as fast as $(x^2+y^2)^{3/2}$ as $x, y \to 0$. More precisely, it means that $|f(x,y) - (ax^2/2 + by^2/2)|/(x^2+y^2)^{3/2}$ is bounded in a neighborhood of the origin. [Note: since everything is smooth, there is a Taylor series for $f(x,y)$. The expression $O(r^3)$ describes all the terms that go as $x^i y^j$ with $i + j \geq 3$. This also means that the derivatives of the $O(r^3)$ terms are $O(r^2)$, and that the second derivatives are $O(r)$.]*

First translate $S_1$ by $-p$, so that the relevant point moves to the origin. Then rotate so that $T_pS$ becomes the $x$-$y$ plane. Then rotate about the $z$-axis so that the principal directions at the origin are along the $x$ and $y$ axes. Use $x$ and $y$ as coordinates, so that the first fundamental form at the origin is the identity matrix. Since $(1,0)$ and $(0,1)$ are eigenvectors of the Weingarten map (whose matrix is the same as the second fundamental form, since the first fundamental form is the identity), the second fundamental form must be diagonal, with $L(0,0) = a$, $M(0,0) = 0$ and $N(0,0) = b$. This gives the second derivatives of $z$ with respect to $x$ and $y$. By Taylor's theorem, $z = ax^2/2 + by^2/2 + O(r^3)$.

*In the rest of this problem set, your answers should all be of the form (Some quantity) = (Some expression involving $a, b, x, y$) + $O(r^{\text{some power}})$. Don't forget that $(1+\epsilon)^n = 1 + n\epsilon + O(\epsilon^2)$. This is particularly useful for $n = 1/2$ and $n = -1$.*

*2. Using coordinates $u = x$ and $v = y$, find expressions for the first and second fundamental forms of $S_2$ as a function of $x, y$, and compute the Gauss curvature $K$.*

Since $\sigma(u,v) = (u, v, f(u,v))$, $\sigma_u = (1, 0, f_u) = (1, 0, ax + O(r^2))$, and $\sigma_v = (0, 1, f_v) = (0, 1, by + O(r^2))$, so the first fundamental form is $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 + a^2x^2 & abxy \\ abxy & 1 + b^2y^2 \end{pmatrix} + O(r^3)$. Meanwhile, $\mathbf{N} = (-f_u, -f_v, 1)/\sqrt{1 + f_u^2 + f_v^2} = (-ax, -by, 1) + O(r^2)$, so the second fundamental form is $\begin{pmatrix} L & M \\ M & N \end{pmatrix} =$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + O(r)$ The Gauss curvature is $(LN - M^2)/(EG - F^2) = ab + O(r)$. Note that it's impossible to get rid of the $O(r)$ correction, since there's no reason to believe that the gradient of $K$ is zero at $p$.

*3. Compute all the Christoffel symbols for $S_2$ (see Prop 7.4.4), and compute the commutator $[\nabla_1, \nabla_2]$. Your answer should be a $2 \times 2$ matrix, from which you can infer the value of $C_1$ (as defined in class).*

Since the first fundamental form is the identity $+O(r^2)$ (as is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$), we have

$$\begin{array}{rcl} \Gamma^1_{11} & = E_u/2 + O(r^2) & = a^2x + O(r^2) \\ \Gamma^2_{11} & = F_u - (E_v/2) + O(r^2) & = aby + O(r^2) \\ \Gamma^1_{12} = \Gamma^1_{21} & = E_v/2 + O(r^2) & = O(r^2) \\ \Gamma^2_{12} = \Gamma^2_{21} & = G_u/2 + O(r^2) & = O(r^2) \\ \Gamma^1_{22} & = F_v - (G_u/2) + O(r^2) & = abx + O(r^2) \\ \Gamma^2_{22} & = G_v/2 + O(r^2) & = b^2y + O(r^2). \end{array}$$

We can package the Christoffel symbols into two matrices:

$$\begin{array}{rcl} A_1 & = & \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^2_{11} & \Gamma^2_{12} \end{pmatrix} = \begin{pmatrix} a^2x & 0 \\ aby & 0 \end{pmatrix} + O(r^2) \\ A_2 & = & \begin{pmatrix} \Gamma^1_{21} & \Gamma^1_{22} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} 0 & abx \\ 0 & b^2y \end{pmatrix} + O(r^2) \end{array}$$

We then have $[\nabla_1, \nabla_2] = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$. However, since $A_1$ and $A_2$ are $O(r)$, $[A_1, A_2] = O(r^2)$, so we have $[\nabla_1, \nabla_2] = \partial_1 A_2 - \partial_2 A_1 + O(r^2) = \begin{pmatrix} 0 & ab \\ -ab & 0 \end{pmatrix} + O(r)$. Since a 90 degree clockwise rotation is $(EG - F^2)^{-1/2} \begin{pmatrix} F & E \\ -G & -F \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O(r^2)$, we infer that $C_1 = ab + O(r)$.

*4. Show that the Gauss equations (Prop. 10.1.2, not to be confused with the Gauss equations of Prop 7.4.4 – Gauss had a lot of equations!) apply to $S_2$ at the origin. An earlier version of this problem also asked about the Codazzi-Mainardi equation. Do NOT evaluate those, as the expressions depend strongly on the $O(r^3)$ terms in $f(x, y)$. Also, this is practically the same calculation as problem 3. Either problem is enough to conclude that $K = C_1$ at the origin.*

Since all of the Christoffel symbols vanish at the origin, the only terms that contribute to the right-hand side are the partial derivatives. Also, $E = G = 1$ and $F = 0$ at the origin. The four equations then become:

$$
\begin{array}{rcl}
EK = K & = & \partial_2\Gamma_{11}^2 - \partial_1\Gamma_{12}^2 = ab - 0 = ab \\
FK = 0 & = & \partial_1\Gamma_{12}^1 - \partial_2\Gamma_{11}^1 = 0 - 0 = 0 \\
FK = 0 & = & \partial_2\Gamma_{12}^2 - \partial_1\Gamma_{22}^2 = 0 - 0 = 0 \\
GK = K & = & \partial_1\Gamma_{22}^1 - \partial_2\Gamma 12^1 = ab - 0 = ab
\end{array}
$$

Since $K = ab$, it works!

*5. Returning to the original surface $S_1$, show that $C_1(p) = K(p)$. Conclude that $C_1$ and $K$ are the same geometric quantity for all points on all surfaces.*

$C_1$ is computed from the first fundamental form, so it does not change under isometries.

If you translate a surface, you don't change any of the vectors $\sigma_u$, $\sigma_v$, $\sigma_{uu}$, $\sigma_{uv}$, $\sigma_{vv}$, so you don't change the first fundamental form or the second fundamental form, so you don't change $K = (LN - M^2)/(EG - F^2)$. If you rotate a surface, you rotate all of the aforementioned vectors, but don't change any of their dot products, and so you don't change the first or second fundamental forms or change $K$. Thus $K$ is invariant under direct isometries. This means that $K(p)$ for $S_1$ equals $K(0)$ for $S_2$. Likewise, $C_1(p)$ for $S_1$ equals $C_1(0)$ for $S_2$. Since $C_1(0) = K$ for $S_2$ (in that they both equal $ab$), $C_1(p) = K(p)$ for $S_1$.

The quantity $K$ does not depend on a choice of coordinates. Neither does $C_2$, which is the rotation per unit area, regardless of coordinates. Since $C_1 = C_2$ (as we showed in class), $C_1$ is independent of coordinates. Thus $C_1(p) = K(p)$ regardless of what coordinates we use. Since $p$ was an arbitrary point, $C_1 = K$ everywhere.

*6. Geodesics on $S_2$ are approximated very well by intersections of $S_1$ with vertical planes through the origin. That is, the shortest path from $(0,0,0)$ to $(x_0, y_0, z_0)$ has $x/y$ constant. Taking this result for granted, compute the distance from the origin to $(x, y, f(x, y))$.*

Work in cylindrical coordinates $(\rho, \theta, z)$, where $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$. Define $g(\theta) = a\cos^2(\theta) + b\sin^2(\theta)$, so our surface is $z = \rho^2 g(\theta)/2 + O(\rho^3)$. Note, by the way, that $g(\theta) = \frac{a+b}{2} + \frac{a-b}{2}\cos(2\theta)$, and that
$g'(\theta) = 2(b-a)\sin(\theta)\cos(\theta) = (b-a)\sin(2\theta)$.

Let $r(\rho, \theta)$ be the distance along a geodesic from the origin to $(x, y, f(x, y))$, which we are approximating with a path of constant $\theta$. Since $\partial z/\partial\rho = \rho g(\theta)$, $dr/d\rho = \sqrt{1 + g^2\rho^2} + O(\rho^3) = 1 + g^2\rho^2/2 + O(\rho^3)$. Integrating we get $r = \rho + g^2\rho^3/6 + O(\rho^4)$.

*7. Using the results of Problem 6, construct geodesic normal coordinates around the origin of $S_2$.*

Our coordinates are now $r$ and $\theta$, and $\rho = r - r^3g^2/6 + O(r^4)$. We then have $\sigma(r, \theta) = (r(1 - r^2g^2/6)\cos(\theta) + O(r^4), r(1 - r^2g^2/6)\sin(\theta) + O(r^4), r^2g/2 + O(r^3))$.

*8. Now consider the "circle" obtained by fixing a value of $r$ in the geodesic normal coordinates. [Note that in this context $r$ is the geodesic distance from the origin, which is NOT the same as $\sqrt{x^2 + y^2}$. This isn't a repeat of an exam problem!] Show that the circumference of that "circle" is $2\pi r(1 - abr^2/6)+$ higher order. This shows that the defect in the circumference is proportional to the Gauss curvature.*

$\sigma_\theta = (-r(1-r^2g^2/6)\sin(\theta) - r^3gg'\cos(\theta)/3 + O(r^4), r(1-r^2g^2/6)\cos(\theta) - r^3gg'\sin(\theta)/3 + O(r^4), r^2g'/2 + O(r^3))$, so $G = r^2(1 - r^2(g^2/3 - (g')^2/4)) + O(r^5)$, as all the $gg'$ terms cancel. Thus $\sqrt{G} = r(1 - r^2(g^2/6 - (g')^2/8)) + O(r^4)$. Integrating over the circle, the average value of $(g')^2$ is $(b-a)^2/2$, and the average value of $g^2 = \frac{(a+b)^2}{4} + \frac{a^2-b^2}{4}\cos(2\theta) + \frac{(a-b)^2}{4}\cos^2(\theta)$ is $(a+b)^2/4 + (a-b)^2/8$. Thus our circumference is

$$2\pi r\left(1 - r^2\left(\frac{(a+b)^2}{24} + \frac{(a-b)^2}{48} - \frac{(a-b)^2}{16}\right)\right) + O(r^4) = 2\pi r(1 - abr^2/6) + O(r^4).$$

*9. Finally, compute the area enclosed by the circle [hint: $\int$ (circumference) $dr$] and the isoperimetric ratio Area/(circumference)$^2$.*

*Your answer to problem 9 SHOULD match the results of the exam problem. In the exam, we considered the intersection of a cylinder $x^2 + y^2 = r^2$ with the surface. That resulted in a curve that approximates the "circle" of problems 8 and 9. That approximation wasn't good enough to compute the circumference and area individually, but WAS good enough to compute the isoperimetric ratio. This is because a circle maximizes the ratio, so deviations from that circular shape only affect the ratio to second order.*

Area $= \int 2\pi r - ab\pi r^3/3 + O(r^4)\, dr = \pi r^2 - \pi abr^4/12 + O(r^5)$
$= \pi r^2(1 - abr^2/12 + O(r^3)) = \pi r^2(1 - Kr^2/12 + O(r^3))$.

Area/(circumference)$^2 = \frac{\pi r^2}{(2\pi r)^2}\frac{1-Kr^2/12+O(r^3)}{(1-Kr^2/6+O(r^3))^2} = \frac{1}{4\pi}(1 + Kr^2/4 + O(r^3))$.