M365G Final Exam, May 14, 2012

1. Cylindrical curves.

a) Consider a curve of the form $\gamma(t) = (a\cos(t), a\sin(t), f(t))$, where a is a constant and f is a smooth function. Compute the speed, curvature and torsion at time t in terms of the (unknown) function f and its derivatives.

b) Now think of this curve as lying on the cylinder $x^2 + y^2 = a^2$, with parametrization $\sigma(u,v) = (a\cos(u), a\sin(u), v)$. Compute the geodesic and normal curvatures in terms of f and its derivatives.

2. Curves as level sets. Prove the following theorem:

Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions on $\mathbb{R}^3$. Suppose that ∇f and ∇g are linearly independent at every point where $f = g = 0$. Then each path component of the level set $\{\mathbf{x} \in \mathbb{R}^3 | f(\mathbf{x}) = g(\mathbf{x}) = 0\}$ is the image of a smooth regular curve.

3. Let γ be a smooth closed curve with curvature bounded below by a positive constant δ. Pick a small positive constant ϵ, and consider the surface with local parametrization $\sigma(u,v) = \gamma(u) + \epsilon\cos(v)\mathbf{n}(u) + \epsilon\sin(v)\mathbf{b}(u)$, where $\mathbf{n}$ and $\mathbf{b}$ are the principal normal and binormal to γ. Show that, for ϵ small enough, this is a regular parametrization. Then give an example to demonstrate why the "for ϵ small enough" condition is needed.

4. Consider the hyperboloid of one sheet, parametrized as follows:
$\sigma(u,v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), \sinh(u))$. Compute the first and second fundamental forms, the principal curvatures, and the Gauss curvature as functions of u and v. (Some of these depend on your choice of normal vector – make clear which one you're using.)

5. Let a be a positive constants Consider the cone $z^2/a^2 = x^2 + y^2$ (restricted to $z > 0$), parametrized as $\sigma(u,v) = (u\cos(v), u\sin(v), au)$. a) Compute the first fundamental forms and all the Christoffel symbols.

b) Show that $[\nabla_u, \nabla_v] = 0$.

c) Compute the effect of parallel transport in the v direction from $v = 0$ to $v = 2\pi$. (Remember that σ_u and σ_v do not have the same size!)

d) Compute the second fundamental form and show directly that the Gauss curvature is zero.

6. Consider a torus obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z axis. (a) Write down a parametrization of this torus.

(b) Find the first fundamental form.

(c) Find three qualitatively different geodesics. (You should explain why they are in fact geodesics, but you don't have to explicitly show that they satisfy the geodesic equations.)