M365G First Midterm Exam, February 16, 2012

1. Prove the following theorem (which we proved in class): *Let* $\mathbf{p} \in \mathbb{R}^2$*, let* U *be an open neighborhood of* $\mathbf{p}$*, and let* $\sigma : U \to \mathbb{R}^3$ *be a smooth surface patch of some surface* S*, and let* $\mathbf{q} = \sigma(\mathbf{p})$*. If* $\sigma_u(\mathbf{p}) \times \sigma_v(\mathbf{p}) \neq 0$*, then there is a neighborhood of* $\mathbf{q}$ *in* S *that can be written as the graph of a smooth function* $f : \mathbb{R}^2 \to \mathbb{R}$*. (That is, either we locally have* $z = f(x, y)$*, or we have* $y = f(x, z)$*, or we have* $x = f(y, z)$*.)*

In your proof, you can make free use of the inverse function theorem for maps $\mathbb{R}^n \to \mathbb{R}^n$, but you shouldn't assume any results about surfaces beyond the basic definitions (unless you prove them, of course).

2. Write down a parametrization for the (planar) ellipse $x^2/9 + y^2/16 = 25$ (with $z = 0$, of course), so that you go around the ellipse counter-clockwise. Then use this parametrization to compute the unit tangent vector $\vec{T}$, the signed normal $\vec{N}_s$ and the signed curvature κ_s at the point $(12, 12)$. [Hint: do NOT attempt to parametrize by arclength, which requires an integral that can't be done in closed form. Do all your computations in terms of t, not s.]

3. Consider the curve $\gamma(t) = (\cos(t), \sin(t), t^3/3)$. Compute the curvature and torsion as a function of t.