M365G Second Midterm Exam, March 22, 2012

1. Surfaces of revolution.

Consider the surface of revolution

$$\sigma(u,\phi) = (r(u)\cos(\phi), r(u)\sin(\phi), g(u)),$$

where $r(u)$ and $g(u)$ are smooth functions, $r(u) > 0$ and $r'(u)^2 + g'(u)^2 \neq 0$. (The book calls r "f", but in problem 2 we're going to use the letter f differently, so let's call it r.)

a) Compute the first fundamental form in terms of the functions r and g. (With ϕ playing the role of the variable that we usually call v.)

b) Fix a value of ϕ_0. Show that the shortest path along the surface from $(u,\phi) = (a,\phi_0)$ to (b,ϕ_0) has ϕ constant.

2. Conformal and equiareal maps. Let $f : S_1 \to S_2$ be a map from a surface of revolution (using the notation of Problem 1) to the right circular cylinder $x^2 + y^2 = 1$, such that $f(\sigma(u,\phi)) = (\cos(\phi), \sin(\phi), h(u))$, for some smooth function $h : \mathbb{R} \to \mathbb{R}$ with $h' \neq 0$.

a) Find conditions on h (in terms of r and g) that make the map f conformal. Find (different!) conditions on h that make the map f equiareal.

b) Suppose that S_1 is the unit sphere, minus the poles, with $r(u) = \cos(u)$ and $g(u) = \sin(u)$ and with $-\pi/2 < u < \pi$. Find a conformal map f to the right circular cylinder. (In cartography, this is called a Mercator projection.) Then apply your criteria from (a) to get an equiareal map (which should yield a very familiar result).

c) Suppose instead that S_1 is the plane minus the origin, with $r(u) = u$, $g(u) = 0$, and with $0 < u < \infty$. Find an explicit conformal map f to the cylinder. (BTW, combining this with the inverse of the conformal map you found in (b) is yet another way to realize stereographic projection.)

3. (2 pages) In this exercise we're going to explore lengths and areas in slightly curved paraboloids. Let S be the surface $z = ax^2 + by^2$, where a and b are small constants, and we are using x and y as our coordinates. (That is, $\sigma(u,v) = (u, v, au^2 + bv^2)$)

a) Compute the first fundamental form.

b) Set up (but don't evaluate) an explicit integral that computes exactly the length of the path $\gamma(t) = \sigma(\cos(t), \sin(t))$ as t goes from 0 to 2π.

c) Set up another explicit integral that computes exactly the area of the surface enclosed by this curve.

d) (Extra credit) Evaluate the integrals from part (b) and (c) to second order in a and b. That is, keep terms like a^2, ab and b^2, but ignore anything of higher order. You may find the identity $\sqrt{1+\alpha} \approx 1 + (\alpha/2)$ (for small α) useful, as well as the integrals $\int_0^{2\pi} \sin^2(\theta)\cos^2(\theta) d\theta = \pi/4$ and $\int\int_D x^2 dx dy = \int\int_D y^2 dx dy = \pi/4$, where D is the unit disk. Then compute the isoperimetric ratio (area)/(length)^2 to second order in a and b.

4. Consider the ruled surface $\sigma(u,v) = \vec{\gamma}(u) + v\vec{\delta}(u)$, where $\gamma(u) = (4\cos(u), 4\sin(u), 0)$, $\delta(u) = (\cos(u)\cos(u/2), \sin(u)\cos(u/2), \sin(u/2))$, u ranges over the entire real line (tracing the same pattern over and over again) and $-1 < v < 1$.

a) Show that this is a well-defined smooth surface.

b) Is this surface orientable? Either prove that it is or prove that it isn't.