M365G Third Midterm Exam, April 12, 2012

1. Graphs.

Consider the surface $z = f(x, y)$, where we take coordinates $u = x$ and $v = y$. Orient the surface so that the normal has positive z-coordinate.

a) Compute the first and second fundamental forms in terms of f and various (partial) derivatives of f.

b) Apply your results from (a) to the paraboloid $z = (x^2 + y^2)/2$ to compute the Gauss curvature K as a function of x and y.

c) Compute $\iint_S K d(area)$ where S is the (unbounded) surface of part (b). [Hint: this can be done directly, but it's easier to think about the image of the Gauss map.]

2. Derive the formulas for the Christoffel symbols. That is, show that

$$\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} E_v/2 \\ G_u/2 \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}$$

3. Consider a tangent developable $\sigma(u, v) = \gamma(u) + v\dot{\gamma}(u)$ with $v > 0$. You can assume that γ is a unit-speed curve and that $\ddot{\gamma}$ is never zero. Compute the first and second fundamental forms as functions of u and v, and show that the Gauss curvature is 0. (We previously showed that a tangent developable is isometric to a subset of a plane. This shows that it has the same Gauss curvature as a plane.)

4. Consider a surface $\sigma(u, v)$ whose first fundamental form is $E = G = 1/v^2$, $F = 0$. The domain of definition has $v > 0$ and u arbitrary.

a) Compute all Christoffel symbols.

b) Write down the equations for a geodesic. (You don't have to actually solve these equations or construct any geodesics)

c) Describe the action of parallel transport of a vector as you move "horizontally" (holding v constant). Then describe the action of parallel transport as you move "vertically" (u constant).

Extra credit: Start with a vector $\mathbf{w} = \sigma_u$ at the point $u = 0$, $v = 1$. Parallel transport this vector counter-clockwise along the "square" whose corners have coordinates (0,1), (1,1), (1,2) and (0,2), varying first u, then v, then u, and then v. By what net angle does the vector rotate? Clockwise or counterclockwise?