M365G First Midterm Exam Solutions, February 16, 2012

1. Prove the following theorem (which we proved in class): *Let* $\mathbf{p} \in \mathbb{R}^2$*, let* U *be an open neighborhood of* $\mathbf{p}$*, and let* $\sigma : U \to \mathbb{R}^3$ *be a smooth surface patch of some surface* S*, and let* $\mathbf{q} = \sigma(\mathbf{p})$*. If* $\sigma_u(\mathbf{p}) \times \sigma_v(\mathbf{p}) \neq 0$*, then there is a neighborhood of* $\mathbf{q}$ *in* S *that can be written as the graph of a smooth function* $f : \mathbb{R}^2 \to \mathbb{R}$*. (That is, either we locally have* $z = f(x, y)$*, or we have* $y = f(x, z)$*, or we have* $x = f(y, z)$*.)*

In your proof, you can make free use of the inverse function theorem for maps $\mathbb{R}^n \to \mathbb{R}^n$, but you shouldn't assume any results about surfaces beyond the basic definitions (unless you prove them, of course).

The basic setup is that we have a map $\sigma : U \to \mathbb{R}^3$. Let $x(u, v)$, $y(u, v)$ and $z(u, v)$ be the three components of $\sigma(u, v)$. Note that $\sigma_u \times \sigma_v = (y_u z_v - y_v z_u, z_u x_v - z_v x_u, x_u y_v - x_v y_u)$, where the subscripts denote partial derivatives with respect to u or v. Since we have assumed that $\sigma_u(p) \times \sigma_v(p) \neq 0$, at least one of those three functions must be nonzero at p.

Suppose that $x_u(p)y_v(p) - x_v(p)y_u(p) \neq 0$. Define a map $g : U \to \mathbb{R}^2$ by $g(u, v) = (x(u, v), y(u, v))$. Then $DG = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$. By assumption, $det(DG) \neq 0$ at p. By the inverse function theorem, there is a smooth function $h : \mathbb{R}^2 \to \mathbb{R}^2$ that is the inverse of G in a neighborhood of $G(p)$. That is, $(u, v) = h(x, y)$. But then $z(u, v) = z(h(x, y))$ is a smooth function of x and y.

(If we had $y_u z_v - y_v z_u \neq 0$, we would have defined $G(u, v) = (y, z)$, and if we had $z_u x_v - z_v x_u \neq 0$ we would have defined $G(u, v) = (x, z)$. In each case, you can write the remaining variable as a smooth function of the other two.)

Note that we do NOT speak of σ^{-1} being smooth. σ^{-1} is only defined on a closed subset of $\mathbb{R}^3$, so it is meaningless to speak of partial derivatives of σ^{-1} with respect to x, y, or z, or of σ^{-1} being smooth. $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ is smooth, in that it has partial derivatives of all orders, $G : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth (being the compososition of the natural projection from $\mathbb{R}^3 \to \mathbb{R}^2$ and σ), and $h : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth (by the inverse function theorem), but σ^{-1} isn't.

Also note that the inverse function theorem is about maps from $\mathbb{R}^n \to \mathbb{R}^n$, not from $\mathbb{R}^2 \to \mathbb{R}^3$, or $\mathbb{R}^3 \to \mathbb{R}^2$, or any other $\mathbb{R}^n \to \mathbb{R}^m$ with $m \neq n$.

2. Write down a parametrization for the (planar) ellipse $x^2/9 + y^2/16 = 25$ (with $z = 0$, of course), so that you go around the ellipse counter-clockwise. Then use this parametrization to compute the unit tangent vector $\vec{T}$, the signed normal $\vec{N}_s$ and the signed curvature κ_s at the point $(12, 12)$. [Hint: do NOT attempt to parametrize by arclength, which requires an integral that can't be done in closed form. Do all your computations in terms of t, not s.]

The simplest periodic parametrization, with period 2π, is $\gamma(t) = (15\cos(t), 20\sin(t))$, which hits $(12, 12)$ when $\cos(t) = 4/5$ and $\sin(t) = 3/5$. We then compute:

$\dot{\gamma}(t) = (-15\sin(t), 20\cos(t))$, $|\dot{\gamma}| = \sqrt{15^2\cos^2(t) + 400\sin^2(t)}$.
For notational simplicity, let $K(t) = 15^2\cos^2(t) + 400\sin^2(t)$, so
$\vec{T}(t) = (-15\sin(t), 20\cos(t))/\sqrt{K(t)}$ and $\vec{N}_s(t) = (-20\cos(t), -15\sin(t))/\sqrt{K(t)}$.

There are two approaches to computing κ_s. One is to note that we're going counter-clockwise, so $\kappa_s = \kappa$, and use the 3D formula $\kappa = |\dot{\gamma} \times \ddot{\gamma}|/|\dot{\gamma}|^3$. Since $\ddot{\gamma} = (-15\cos(t), -20\sin(t))$, the cross product (in $\mathbb{R}^3$) is $\dot{\gamma} \times \ddot{\gamma} = (0, 0, 300)$, hence $\kappa_s = \kappa = 300/K^{3/2}$. Evaluating at $\cos(t) = 4/5$, $\sin(t) = 3/5$ we get $K = 256 + 81 = 337$, so $\vec{T} = (-9, 16)/\sqrt{337}$, $\vec{N}_s = (-16, -9)/\sqrt{337}$, and $\kappa_s = 300/(337)^{3/2}$.

The second approach is to compute $\kappa_s = \vec{N}_s \cdot \frac{d\vec{T}}{ds} = \vec{N}_s \cdot \frac{d\vec{T}}{dt}/|\dot{\gamma}|$. Taking a few derivatives and doing a bunch of algebra yields $d\vec{T}/dt = 300N_s/K$, hence $\kappa_s = 300K^{-3/2}$.

3. Consider the curve $\gamma(t) = (\cos(t), \sin(t), t^3/3)$. Compute the curvature and torsion as a function of t.

We have $\dot{\gamma}(t) = (-\sin(t), \cos(t), t^2)$, $\ddot{\gamma} = (-\cos(t), -\sin(t)2t)$ and $\dddot{\gamma} = (\sin(t), -\cos(t), 2)$. We compute $|\dot{\gamma}| = \sqrt{1 + t^4}$, and $\dot{\gamma} \times \ddot{\gamma} = (2t\cos(t) + t^2\sin(t), 2t\sin(t) - t^2\cos(t), 1)$, so $|\dot{\gamma} \times \ddot{\gamma}| = \sqrt{1 + 4t^2 + t^4}$ and $(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma} = 2 + t^2$. As a result:

$\kappa = |\dot{\gamma} \times \ddot{\gamma}|/|\dot{\gamma}|^3 = \sqrt{1 + 4t^2 + t^4}(1 + t^4)^{-3/2}$ and $\tau = (\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}/|\dot{\gamma} \times \ddot{\gamma}|^2 = (2 + t^2)/(1 + 4t^2 + t^4)$.