M408S Homework 7. Due Monday, March 4

Part 1: Improper integrals

1) A car is driving along a straight road at 1 mile/minute. An observer on a hill a mile from the road is tracking the progress of the car. Let P be the point on the road closest to the observer. The observer turns so that he is always facing the car.

a) At what rate is the observer rotating when the car is x miles away from P?

b) What is the total rotation of the observer, pretending that the car has been driving forever and will continue driving forever, with the observer always tracking the car? (Obviously, in real life there are finite start and end times.)

c) What is $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$? What does this have to do with parts (a) and (b)?

2) What is wrong with the following calculation? Explain.

$$\int_{-1}^{2} \frac{dx}{x^2} = \frac{-1}{x}\Big|_{-1}^{2} = \frac{-1}{2} - (-1) = \frac{1}{2}.$$

3) In most cases, u-substitutions for improper integrals work the same as with ordinary integrals. Use the substitution $u = \ln(x)$ to convert the integrals $\int_e^{\infty} \frac{dx}{x(\ln(x))^2}$ and $\int_1^{e} \frac{dx}{x\sqrt{\ln(x)}}$ to easier integrals, and evaluate them.

4) For what values of p does the integral $\int_e^{\infty} \frac{dx}{x(\ln(x))^p}$ converge? For what values of p does $\int_1^{e} \frac{dx}{x(\ln(x))^p}$ converge?

5) Compute $\int_0^{\infty} x^n e^{-x} dx$ for $n = 0, 1, 2$ and 3. Do you see a pattern? Can you find a formula that works for all positive integer values of n? [Hint: integrate by parts.]

Part 2: Separable differential equations

6) Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2.$$

Sketch at least two particular solutions. How do they differ?

7) Find the general solution to the differential equation

$$\frac{dy}{dx} = y^2.$$

Sketch at least two particular solutions. How do they differ?

8) Find the general solution to the differential equation

$$\frac{dy}{dx} = xe^{-y}.$$

(You do not have to sketch this one.)