M382D Final Exam, May 6, 2016

Do **three** out of four problems. (The fourth is on the back.) Be clear about which problems you want graded. If you attempt all four problems, I'll just grade the first three.

1) The *Hopf fibration* is the quotient map $\pi : S^3 \to CP^1$, where we think of S^3 as the unit sphere in $\mathbb{C}^2$ and the quotient is by the equivalence relation $(z_1, z_2) \sim (e^{i\theta}z_1, e^{i\theta}z_2)$ for arbitrary θ and arbitrary $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$.

Prove that there does not exist a smooth map $s : CP^1 \to S^3$ such that $\pi \circ s$ is the identity on CP^1. [Hint: If $Z = \pi^{-1}(p)$ for your favorite point $p \in CP^1$, what is $I(s, Z)$? This problem can also be solved using cohomology, but IMO it's easier to use intersection theory.]

2) Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Consider the maps $X \to X$ induced by the following maps $\mathbb{R}^2 \to \mathbb{R}^2$. For each one, (i) find the fixed points on X, (ii) compute the Lefschetz number, and (iii) either prove that the map is homotopic to the identity on X or prove that it isn't.

a) $f(x, y) = (y, -x)$

b) $g(x, y) = (y, x)$.

3) A *knot* is an embedding: $K : S^1 \to \mathbb{R}^3$, where for definiteness we take $S^1 = \mathbb{R}/\mathbb{Z}$. A *link* L is a pair (K, K') of knots such that K and K' do not intersect. That is, for all (u, v), $K(u) \neq K'(v)$. In this exercise we are going to define the *linking number* of L and show that it is a topological invariant.

On $\mathbb{R}^3 - \{0\}$, define the 2-form

$$\omega = \frac{i_x \det}{4\pi|x|^3} = \frac{x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2}{4\pi|x|^3}.$$

You can take as given that $d\omega = 0$ and that $\int_{S^2} \omega = 1$. If $T = S^1 \times S^1$, then a link L defines a map $f : T \to \mathbb{R}^3 - \{0\}$ by $f(u, v) = K'(v) - K(u)$. Define $Link(L) = \int_{T^2} f^*\omega$.

a) Show that $Link(L)$ is a homotopy invariant. That is, if we homotope K and K' so that at all times (K_t, K'_t) remains a link, then $Link(K_1, K'_1) = Link(K_0, K'_0)$.

b) Find an explicit integral formula for $Link(L)$

4)Let ω be as in Probelm 3. Suppose that $D \subset \mathbb{R}^3$ is a compact 3-manifold-with-boundary. Let $f : D \to \mathbb{R}^3$ be a smooth function such that $0 \notin f(\partial D)$. Show that if $\int_{\partial D} f^*\omega \neq 0$, then $0 \in f(D)$. [Hint: This can be proved directly using properties of forms, or indirectly by relating the integral to a winding number.]