M382D Final Exam Solutions, May 6, 2016

Do **three** out of four problems. (The fourth is on the back.) Be clear about which problems you want graded. If you attempt all four problems, I'll just grade the first three.

1) The *Hopf fibration* is the quotient map $\pi : S^3 \to CP^1$, where we think of S^3 as the unit sphere in $\mathbb{C}^2$ and the quotient is by the equivalence relation $(z_1, z_2) \sim (e^{i\theta} z_1, e^{i\theta} z_2)$ for arbitrary θ and arbitrary $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$.

Prove that there does not exist a smooth map $s : CP^1 \to S^3$ such that $\pi \circ s$ is the identity on CP^1. [Hint: If $Z = \pi^{-1}(p)$ for your favorite point $p \in CP^1$, what is $I(s, Z)$? This problem can also be solved using cohomology, but IMO it's easier to use intersection theory.]

If s is such a map, then $I(s, Z) = \pm 1$, since there is a unique intersection points at $s(p)$, and the intersection is clearly transverse, since TZ is the kernel of $d\pi$ and $d\pi \circ ds$ is the identity on TCP^1. However, s is homotopic to a constant map, since s isn't onto (by Sard) and all maps to S^3 minus a point are homotopic to constants. (This was a homework exercise, I believe, to prove that intersection theory on S^n is trivial except when one of the factors is n-dimensional.) So $I(s, Z) = 0$, which is a contradiction.

Another approach is to say that $\pi \circ s$ is the identity on CP^1, so $s^\sharp \circ \pi^\sharp$ must be the identity on $H^2(CP^1)$. However, $\pi^\sharp$ cannot be injective, since $H^2(S^3) = 0$ while $H^2(CP^1) = \mathbb{R}$. Contradiction.

2) Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Consider the maps $X \to X$ induced by the following maps $\mathbb{R}^2 \to \mathbb{R}^2$. For each one, (i) find the fixed points on X, (ii) compute the Lefschetz number, and (iii) either prove that the map is homotopic to the identity on X or prove that it isn't.

a) $f(x, y) = (y, -x)$
b) $g(x, y) = (y, x)$.

(a) This map has exactly two fixed points, namely $(0, 0)$ and $(1/2, 1/2)$. At each one, $\det(df - I) = 2 > 0$, so $L(f) = 2$. Since $L(f) \neq \chi(X)$, f cannot be homotopic to the identity.

(b) Now the fixed points are the entire diagonal $y = x$. To compute the Lefschetz number, we homotope this to $\tilde{g}(x, y) = (y + 0.1, x + 0.1)$, which has no fixed points at all, so $L(g) = 0$. Although the Lefschetz number equals $\chi(X)$, the map is still not homotopic to the identity, since the map is orientation-reversing and has degree -1.

3) A *knot* is an embedding: $K : S^1 \to \mathbb{R}^3$, where for definiteness we take $S^1 = \mathbb{R}/\mathbb{Z}$. A *link* L is a pair (K, K') of knots such that K and K' do not intersect. That is, for all (u, v), $K(u) \neq K'(v)$. In this exercise we are going to define the *linking number* of L and show that it is a topological invariant.

On $\mathbb{R}^3 - \{0\}$, define the 2-form

$$\omega = \frac{i_x \det}{4\pi|x|^3} = \frac{x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2}{4\pi|x|^3}.$$

You can take as given that $d\omega = 0$ and that $\int_{S^2} \omega = 1$. If $T = S^1 \times S^1$, then a link L defines a map $f : T \to \mathbb{R}^3 - \{0\}$ by $f(u, v) = K'(v) - K(u)$. Define $Link(L) = \int_{T^2} f^*\omega$.

a) Show that $Link(L)$ is a homotopy invariant. That is, if we homotope K and K' so that at all times (K_t, K'_t) remains a link, then $Link(K_1, K'_1) = Link(K_0, K'_0)$.

b) Find an explicit integral formula for $Link(L)$

We proved (certainly in homework, and possibly in class as well) that if ω is a closed form on Y and $f_0 \sim f_1$ are homotopic maps $X \to Y$, then $\int_X f_0^*\omega = \int_X f_1^*\omega$. (This is basically Stokes Theorem applied to $F^*\omega$ on $[0, 1] \times X$.) Homotoping the pair (K, K') without allowing intersection results in homotoping the map $f : T \to \mathbb{R}^3 - \{0\}$. Since ω is closed on $\mathbb{R}^3 - \{0\}$, the homework exercise shows that the linking number is a homotopy invariant. In fact, $Link(K, K')$ is the degree of the map f, and degree is already known to be a homotopy invariant.

Doing a little calculus, we get

$$Link(K, K') = \int_0^1 \int_0^1 \frac{\det\left(K(s) - K'(t), \partial_s K(s), \partial_t K'(t)\right)}{4\pi|K(s) - K'(t)|^3} ds\, dt$$

If you prefer, you can rewrite the triple product of $K(s) - K'(t)$, $\partial_s K(s)$, and $\partial_t K(t)$ in terms of dot products and cross products in several different ways, so the are a number of different-looking versions of this answer.

4)Let ω be as in Probelm 3. Suppose that $D \subset \mathbb{R}^3$ is a compact 3-manifold-with-boundary. Let $f : D \to \mathbb{R}^3$ be a smooth function such that $0 \notin f(\partial D)$. Show that if $\int_{\partial D} f^*\omega \neq 0$, then $0 \in f(D)$. [Hint: This can be proved directly using properties of forms, or indirectly by relating the integral to a winding number.]

If there are no zeroes of f in D, then $f^*\omega$ is a well-defined 2-form on D. However, $d(f^*\omega) = f^*(d\omega) = 0$, so by Stokes' Theorem,

$$0 = \int_D d(f^*\omega) = \int_{\partial D} f^*\omega \neq 0.$$

Contradiction.