NOTES ON DIFFERENTIAL FORMS. PART 2: STOKES' THEOREM

1. Stokes' Theorem on Euclidean Space

Let $X = H^n$, the half space in $\mathbb{R}^n$. Specifically, $X = \{x \in \mathbb{R}^n | x_n \geq 0\}$. Then ∂X, viewed as a set, is the standard embedding of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$. However, the orientation on ∂X is *not* necessarily the standard orientation on $\mathbb{R}^{n-1}$. Rather, it is $(-1)^n$ times the standard orientation on $\mathbb{R}^{n-1}$, since it take $n-1$ flips to change $(n, e_1, e_2, \ldots, e_{n-1}) = (-e_n, e_1, \ldots, e_{n-1})$ to $(e_1, \ldots, e_{n-1}, -e_n)$, which is a negatively oriented basis for $\mathbb{R}^n$.

Theorem 1.1 (Stokes' Theorem, Version 1)**.** *Let ω be any compactly-supported $(n-1)$-form on X. Then*

$$\int_X d\omega = \int_{\partial X} \omega. \tag{1}$$

Proof. Let I_j be the ordered subset of $\{1, \ldots, n\}$ in which the element j is deleted. Suppose that the $(n-1)$-form ω can be expressed as $\omega(x) = \omega_j(x) dx^{I_j}$, where $\omega_j(x)$ is a compactly supported function. Then $d\omega = (-1)^{j-1} \partial_j \omega_j dx^1 \wedge \cdots \wedge dx^n$. There are two cases to consider:

If $j < n$, then the restriction to ∂X of ω is zero, since $dx^n = 0$, so $\int_{\partial X} \omega = 0$. But then

$$\int_{H^n} \partial_j \omega_j dx^1 \cdots dx^n = \int \left[\int_{-\infty}^{\infty} \partial_j \omega_j(x)\, dx_j \right] dx^1 \cdots dx^{j-1} dx^{j+1} \cdots dx^n$$

Since ω_j is compactly supported, the inner integral is zero by the fundamental theorem of calculus. Both sides of (1) are then zero, and the theorem holds.

If $j = n$, then

$$\begin{aligned} \int_X d\omega &= \int_{H^n} (-1)^{n-1} \partial_n \omega_n(x)\, d^n x \\ &= \int_{\mathbb{R}^{n-1}} \left[\int_0^{\infty} (-1)^{n-1} \partial_n \omega_n(x_1, \ldots, x_n)\, dx^n \right] dx^1 \cdots dx^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} (-1)^n \omega_n(x_1, \ldots, x_{n-1}, 0) dx^1 \cdots dx^{n-1} \\ &= \int_{\partial X} \omega. \end{aligned} \tag{2}$$

Here we have used the fundamental theorem of calculus and the fact that ω_n is compactly supported to get $\int_0^{\infty} \partial_n \omega_n(x) dx^n = -\omega_n(x^1, \ldots, x^{n-1}, 0)$.

Of course, not every $(n-1)$-form can be written as $\omega_j dx^{I_j}$ with ω_j compactly supported. However, every compactly-supported $(n-1)$-form can be written as a finite *sum* of such

Date: April 8, 2016.

terms, one for each value of j. Since equation (1) applies to each term in the sum, it also applies to the total. □

The amazing thing about this proof is how easy it is! The only analytic ingredients are Fubini's Theorem (which allows us to first integrate over x^j and then over the other variables) and the 1-dimensional Fundamental Theorem of Calculus. The hard work came earlier, in developing the appropriate definitions of forms and integrals.

2. Stokes' Theorem on Manifolds

Having so far avoided all the geometry and topology of manifolds by working on Euclidean space, we now turn back to working on manifolds. Thanks to the properties of forms developed in the previous set of notes, everything will carry over, giving us

Theorem 2.1 (Stokes' Theorem, Version 2)**.** *Let X be a compact oriented n-manifold-with-boundary, and let ω be an $(n-1)$-form on X. Then*

$$\int_X d\omega = \int_{\partial X} \omega, \tag{3}$$

where ∂X is given the boundary orientation and where the right hand side is, strictly speaking, the integral of the pullback of ω to ∂X by the inclusion map.

Proof. Using a partition-of-unity, we can write ω as a finite sum of forms ω_i, each of which is compactly supported within a single coordinate patch. To spell that out,

- Every point has a coordinate neighborhood.
- Since X is compact, a finite number of such neighborhoods cover X.
- Pick a partition of unity $\{\rho_i\}$ subordinate to this cover.
- Let $\omega_i = \rho_i\omega$. Since $\sum_i \rho_i = 1$, $\omega = \sum_i \omega_i$.

Now suppose that the support of ω_i is contained in the image of an orientation-preserving parametrization $\psi_i : U_i \to X$, where U_i is an open set in H^n. But then

$$\begin{aligned} \int_X d\omega_i &= \int_{\psi(U_i)} d\omega_i \\ &= \int_{U_i} \psi_i^*(d\omega_i) \\ &= \int_{U_i} d(\psi^*\omega_i) \\ &= \int_{H^n} d(\psi^*\omega_i) \\ &= \int_{\partial H^n} \psi^*\omega_i \\ &= \int_{\partial X} \omega_i, \end{aligned} \tag{4}$$

where we have used (a) the definition of integration of forms on manifolds, (b) the fact that d commutes with pullbacks, (c) the fact that $d(\psi^*\omega_i)$ can be extended by zero to all of H^n, (d) Stokes' Theorem on H^n, and (e) the definition of integration of forms on manifolds. Finally, we add everything up.

$$\begin{aligned}
\int_X d\omega &= \int_X d\sum_i \omega_i \\
&= \sum_i \int_X d\omega_i \\
&= \sum_i \int_{\partial X} \omega_i \\
&= \int_{\partial X} \sum_i \omega_i \\
&= \int_{\partial X} \omega. \qquad (5)
\end{aligned}$$

□

Note that X being compact is *essential.* If X isn't compact, then you can still prove Stokes' Theorem for forms that are compactly supported, but not for forms in general. For instance, if $X = [0, \infty)$ and $\omega = 1$ (a 0-form), then $\int_X d\omega = 0$ but $\int_{\partial X} \omega = -1$.

To relate Stokes' Theorem for forms and manifolds to the classical theorems of vector calculus, we need a correspondence between line integrals, surface integrals, and integrals of form.

Exercise 1 If γ is an oriented path in $\mathbb{R}^3$ and $\vec{v}(x)$ is a vector field, show that $\int_\gamma \omega^1_{\vec{v}}$ is the line integral $\int \vec{v} \cdot T ds$, where T is the unit tangent to the curve and ds is arclength measure.

Exercise 2 If S is an oriented surface in $\mathbb{R}^3$ and $\vec{v}$ is a vector field, show that $\int_S \omega^2_{\vec{v}}$ is the flux of $\vec{v}$ through S.

Exercise 3 Suppose that X is a compact connected oriented 1-manifold-with-boundary in $\mathbb{R}^n$. (In other words, a path from a to b, where a and b might be the same point.) Show that Stokes' Theorem, applied to X, is essentially the Fundamental Theorem of Calculus.

Exercise 4 Now suppose that X is a bounded domain in $\mathbb{R}^2$. Write down Stokes' Theorem in this setting and relate it to the classical Green's Theorem.

Exercise 5 Now suppose that S is an oriented surface in $\mathbb{R}^3$ with boundary curve $C = \partial S$. Let $\vec{v}$ be a vector field. Apply Stokes Theorem to $\omega^1_{\vec{v}}$ and to S, and express the result in terms of line integrals and surface integrals. This should give you the classical Stokes' Theorem.

Exercise 6 On $\mathbb{R}^3$, let $\omega = (x^2 + y^2)dx \wedge dy + (x + ye^z)dy \wedge dz + e^x dx \wedge dz$. Compute $\int_S \omega$, where S is the upper hemisphere of the unit sphere. The answer depends on which orientation you pick for S of course. Pick one, and compute! [Find an appropriate surface S' so that $S - S'$ is the boundary of a 3-manifold. Then use Stokes' Theorem to relate $\int_S \omega$ to $\int_{S'} \omega$.]

Exercise 7 On $\mathbb{R}^2$ with the origin removed, let $\alpha = (xdy - ydx)/(x^2 + y^2)$. You previously showed that $d\alpha = 0$ (aka "α is closed"). Show that α is not d of any function ("α is not exact")

Exercise 8 On $\mathbb{R}^3$ with the origin removed, show that $\beta = (xdy \wedge dz - ydx \wedge dz + zdx \wedge dy)/(x^2 + y^2 + z^2)^{3/2}$ is closed but not exact.

Exercise 9 Let X be a compact oriented n-manifold (without boundary), let Y be a manifold, and let ω be a closed n-form on Y. Suppose that f_0 and f_1 are homotopic maps $X \to Y$. Show that $\int_X f_0^* \omega = \int_X f_1^* \omega$.

Exercise 10 Let $f : S^1 \to \mathbb{R}^2 - \{0\}$ be a smooth map whose winding number around the origin is k. Show that $\int_{S^1} f^* \alpha = 2\pi k$, where α is the form of Exercise 7.