NOTES ON DIFFERENTIAL FORMS. PART 3: TENSORS

1. What is a tensor?

Let V be a finite-dimensional vector space.[1] It could be $\mathbb{R}^n$, it could be the tangent space to a manifold at a point, or it could just be an abstract vector space. A *k-tensor* is a map

$$T : V \times \cdots \times V \to \mathbb{R}$$

(where there are k factors of V) that is linear in each factor.[2] That is, for fixed $\vec{v}_2, \ldots, \vec{v}_k$, $T(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}, \vec{v}_k)$ is a linear function of $\vec{v}_1$, and for fixed $\vec{v}_1, \vec{v}_3, \ldots, \vec{v}_k$, $T(\vec{v}_1, \ldots, \vec{v}_k)$ is a linear function of $\vec{v}_2$, and so on. The space of k-tensors on V is denoted $\mathcal{T}^k(V^*)$.

Examples:

- If $V = \mathbb{R}^n$, then the inner product $P(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w}$ is a 2-tensor. For fixed $\vec{v}$ it's linear in $\vec{w}$, and for fixed $\vec{w}$ it's linear in $\vec{v}$.
- If $V = \mathbb{R}^n$, $D(\vec{v}_1, \ldots, \vec{v}_n) = \det \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$ is an n-tensor.
- If $V = \mathbb{R}^n$, $Three(\vec{v}) =$ "the 3rd entry of $\vec{v}$" is a 1-tensor.
- A 0-tensor is just a number. It requires no inputs at all to generate an output.

Note that the definition of tensor says *nothing* about how things behave when you rotate vectors or permute their order. The inner product P stays the same when you swap the two vectors, but the determinant D changes sign when you swap two vectors. Both are tensors. For a 1-tensor like $Three$, permuting the order of entries doesn't even make sense!

Let $\{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for V. Every vector $\vec{v} \in V$ can be uniquely expressed as a linear combination:

$$\vec{v} = \sum_i v^i \vec{b}_i,$$

where each v^i is a number. Let $\phi^i(\vec{v}) = v^i$. The map ϕ^i is manifestly linear (taking $\vec{b}_i$ to 1 and all the other basis vectors to zero), and so is a 1-tensor. In fact, the ϕ^i's form a basis for the space of 1-tensors. If α is any 1-tensor, then

$$\begin{aligned} \alpha(\vec{v}) &= \alpha(\sum_i v^i \vec{b}_i) \\ &= \sum_i v^i \alpha(\vec{b}_i) \qquad \text{by linearity} \end{aligned}$$

Date: April 15, 2016.

[1] Or even an infinite-dimensional vector space, if you apply appropriate regularity conditions.

[2] Strictly speaking, this is what is called a *contravariant* tensor. There are also covariant tensors and tensors of mixed type, all of which play a role in differential geometry. But for understanding forms, we only need contravariant tensors.

$$= \sum_i \alpha(\vec{b}_i)\phi^i(\vec{v}) \quad \text{since } v^i = \phi^i(\vec{v})$$
$$= (\sum_i \alpha(\vec{b}_i)\phi^i)(\vec{v}) \quad \text{by linearity, so}$$
$$\alpha = \sum_i \alpha(\vec{b}_i)\phi^i. \tag{1}$$

A bit of terminology: The space of 1-tensors is called the *dual space* of V and is often denoted V^*. The basis $\{\phi^i\}$ for V^* is called the *dual basis* of $\{b_j\}$. Note that

$$\phi^i(\vec{b}_j) = \delta^i_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

and that there is a duality between vectors and 1-tensors (also called co-vectors).

$$\vec{v} = \sum v^i \vec{b}_i \qquad \text{where } v^i = \phi^i(\vec{v})$$
$$\alpha = \sum \alpha_j \phi^j \qquad \text{where } \alpha_j = \alpha(\vec{b}_j)$$
$$\alpha(\vec{v}) = \sum \alpha_i v^i.$$

It is sometimes convenient to express vectors as columns and co-vectors as rows. The basis vector $\vec{b}_i$ is represented by a column with a 1 in the i-th slot and 0's everywhere else, while ϕ^j is represented by a row with a 1 in the jth slot and the rest zeroes. Unfortunately, representing tensors of order greater than 2 visually is difficult, and even 2-tensors aren't properly described by matrices. To handle 2-tensors or higher, you really need indices.

If α is a k-tensor and β is an ℓ-tensor, then we can combine them to form a $k+\ell$ tensor that we denote $\alpha \otimes \beta$ and call the *tensor product* of α and β:

$$(\alpha \otimes \beta)(\vec{v}_1, \ldots, \vec{v}_{k+\ell}) = \alpha(\vec{v}_1, \ldots, \vec{v}_k)\beta(\vec{v}_{k+1}, \ldots, \vec{v}_{k+\ell}).$$

For instance,

$$(\phi^i \otimes \phi^j)(\vec{v}, \vec{w}) = \phi^i(\vec{v})\,\phi^j(\vec{w}) = v^i w^j.$$

Not only are the $\phi^i \otimes \phi^j$'s 2-tensors, but they form a basis for the space of 2-tensors. The proof is a generalization of the description above for 1-tensors, and a specialization of the following exercise.

Exercise 1: For each ordered k-index $I = \{i_1, \ldots, i_k\}$ (where each number can range from 1 to n), let $\tilde{\phi}^I = \phi^{i_1} \otimes \phi^{i_2} \otimes \cdots \otimes \phi^{i_k}$. Show that the $\tilde{\phi}^I$'s form a basis for $\mathcal{T}^k(V^*)$, which thus has dimension n^k. [Hint: If α is a k-tensor, let $\alpha_I = \alpha(\vec{b}_{i_1}, \ldots, \vec{b}_{i_k})$. Show that $\alpha = \sum_I \alpha_I \tilde{\phi}^I$. This implies that the $\tilde{\phi}^I$'s span $\mathcal{T}^k(V^*)$. Use a separate argument to show that the $\tilde{\phi}^I$'s are linearly independent.]

Among the k-tensors, there are some that have special properties when their inputs are permuted. For instance, the inner product is symmetric, with $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$, while the determinant is anti-symmetric under interchange of any two entries. We can always decompose a tensor into pieces with distinct symmetry.

For instance, suppose that α is an arbitrary 2-tensor. Define

$$\alpha_+(\vec{v},\vec{w}) = \frac{1}{2}\left(\alpha(\vec{v},\vec{w}) + \alpha(\vec{w},\vec{v})\right); \qquad \alpha_-(\vec{v},\vec{w}) = \frac{1}{2}\left(\alpha(\vec{v},\vec{w}) - \alpha(\vec{w},\vec{v})\right).$$

Then α_+ is symmetric, α_- is anti-symmetric, and $\alpha = \alpha_+ + \alpha_-$.

2. Alternating Tensors

Our goal is to develop the theory of differential forms. But k-forms are made for integrating over k-manifolds, and integration means measuring volume. So the k-tensors of interest should behave qualitatively like the determinant tensor on $\mathbb{R}^k$, which takes k vectors in $\mathbb{R}^k$ and returns the (signed) volume of the parallelpiped that they span. In particular, it should change sign whenever two arguments are interchanged.

Let S_k denote the group of permutations of $(1,\ldots,k)$. A typical element will be denoted $\sigma = (\sigma_1,\ldots,\sigma_k)$. The *sign* of σ is $+1$ if σ is an even permutation, i.e. the product of an even number of transpositions, and -1 if σ is an odd permutation.

We say that a k-tensor α is *alternating* if, for any $\sigma \in S_k$ and any (ordered) collection $\{\vec{v}_1,\ldots,\vec{v}_k\}$ of vectors in V,

$$\alpha(\vec{v}_{\sigma_1},\ldots,\vec{v}_{\sigma_k}) = \text{sign}(\sigma)\alpha(\vec{v}_1,\ldots,v_k).$$

The space of alternating k-tensors on V is denoted $\Lambda^k(V^*)$. Note that $\Lambda^1(V^*) = \mathcal{T}^1(V^*) = V^*$ and that $\Lambda^0(V^*) = \mathcal{T}^0(V^*) = \mathbb{R}$.

If α is an arbitrary k-tensor, we define

$$Alt(\alpha) = \frac{1}{k!}\sum_{\sigma\in S_k} \text{sign}(\sigma)\alpha\circ\sigma,$$

or more explicitly

$$Alt(\alpha)(\vec{v}_1,\ldots,\vec{v}_k) = \frac{1}{k!}\sum_{\sigma\in S_k} \text{sign}(\sigma)\alpha(\vec{v}_{\sigma_1},\ldots,\vec{v}_{\sigma_k}).$$

Exercise 2: (in three parts)

(1) Show that $Alt(\alpha) \in \Lambda^k(V^*)$.
(2) Show that Alt, restricted to $\Lambda^k(V^*)$, is the identity. Together with (1), this implies that Alt is a projection from $\mathcal{T}^k(V^*)$ to $\Lambda^k(V^*)$.
(3) Suppose that α is a k-tensor with $Alt(\alpha) = 0$ and that β is an arbitrary ℓ-tensor. Show that $Alt(\alpha\otimes\beta) = 0$.

Finally, we can define a product operation on alternating tensors. If $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^\ell(V^*)$, define

$$\alpha\wedge\beta = C_{k,\ell}\, Alt(\alpha\otimes\beta),$$

where $C_{k,\ell}$ is an appropriate constant that depends only on k and ℓ.

Exercise 3: Suppose that $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^\ell(V^*)$, and that $C_{k,\ell} = C_{\ell,k}$ Show that $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$. In other words, wedge products for alternating tensors have the same symmetry properties as wedge products of forms.

Unfortunately, there are two different conventions for what the constants $C_{k,\ell}$ should be!

(1) Most authors, including Spivak, use $C_{k,\ell} = \dfrac{(k+\ell)!}{k!\ell!} = \dbinom{k+\ell}{k}$. The advantage of this convention is that $\det = \phi^1 \wedge \cdots \wedge \phi^n$. The disadvantage of this convention is that you have to keep track of a bunch of factorials when doing wedge products.
(2) Some authors, including Guillemin and Pollack, use $C_{k,\ell} = 1$. This keeps that algebra of wedge products simple, but has the drawback that $\phi^1 \wedge \cdots \wedge \phi^n(\vec{b}_1, \ldots, \vec{b}_n) = 1/n!$ instead of 1. The factorials then reappear in formulas for volume and integration.
(3) My personal preference is to use $C_{k,\ell} = \dbinom{k+\ell}{k}$, and that's what I'll do in the rest of these notes. So be careful when transcribing formulas from Guillemin and Pollack, since they may differ by some factorials!

Exercise 4: Show that, for both conventions, $C_{k,\ell} C_{k+\ell,m} = C_{\ell,m} C_{k,\ell+m}$.

Exercise 5: Suppose that the constants $C_{k,\ell}$ are chosen so that $C_{k,\ell} C_{k+\ell,m} = C_{\ell,m} C_{k,\ell+m}$, and suppose that α, β and γ are in $\Lambda^k(V^*)$, $\Lambda^\ell(V^*)$ and $\Lambda^m(V^*)$, respectively. Show that

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

[If you get stuck, look on page 156 of Guillemin and Pollack].

Exercise 6: Using the convention $C_{k,\ell} = \dfrac{(k+\ell)!}{k!\ell!} = \dbinom{k+\ell}{k}$, show that $\phi^{i_1} \wedge \cdots \wedge \phi^{i_k} = k! Alt(\phi^{i_1} \otimes \cdots \otimes \phi^{i_k})$. (If we had picked $C_{k,\ell} = 1$ as in Guillemin and Pollack, we would have gotten the same formula, only without the factor of $k!$.)

Let's take a step back and see what we've done.

- Starting with a vector space V with basis $\{\vec{b}_i\}$, we created a vector space $V^* = \mathcal{T}^1(V^*) = \Lambda^1(V^*)$ with dual basis $\{\phi^j\}$.
- We defined an associative product $\wedge$ with the property that $\phi^j \wedge \phi^i = -\phi^i \wedge \phi^j$ and with no other relations.
- Since tensor products of the ϕ^j's span $\mathcal{T}^k(V^*)$, wedge products of the ϕ^j's must span $\Lambda^k(V^*)$. In other words, $\Lambda^k(V^*)$ is exactly the space that you get by taking formal products of the ϕ^i's, subject to the anti-symmetry rule.
- That's *exactly* what we did with the formal symbols dx^j to create differential forms on $\mathbb{R}^n$. The only difference is that the coefficients of differential forms are functions rather than real numbers (and that we have derivative and pullback operations on forms).

- Carrying over our old results from wedges of dx^i's, we conclude that $\Lambda^k(V^*)$ had dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and basis $\phi^I := \phi^{i_1} \wedge \cdots \wedge \phi^{i_k}$, where $I = \{i_1, \ldots, i_k\}$ is an arbitrary subset of $(1, \ldots, n)$ (with k distinct elements) *placed in increasing order.*
- Note the difference between $\tilde{\phi}^I = \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}$ and $\phi^I = \phi^{i_1} \wedge \cdots \wedge \phi^{i_k}$. The tensors $\tilde{\phi}^I$ form a basis for $\mathcal{T}^k(V^*)$, while the tensors ϕ^I form a basis for $\Lambda^k(V^*)$. They are related by $\phi^I = k! Alt(\tilde{\phi}^I)$.

Exercise 7: Let $V = \mathbb{R}^3$ with the standard basis, and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(x, y, z) = (x, y)$ be the projection onto the x-y plane. Let $\alpha(\vec{v}, \vec{w})$ be the signed area of the parallelogram spanned by $\pi(\vec{v})$ and $\pi(\vec{w})$ in the x-y plane. Similarly, let β and γ be be the signed areas of the projections of $\vec{v}$ and $\vec{w}$ in the x-z and y-z planes, respectively. Express α, β and γ as linear combinations of $\phi^i \wedge \phi^j$'s. [Hint: If you get stuck, try doing the next two exercises and then come back to this one.]

Exercise 8: Let V be arbitrary. Show that $(\phi^{i_1} \wedge \cdots \wedge \phi^{i_k})(\vec{b}_{j_1}, \ldots, \vec{b}_{j_k})$ equals $+1$ if $(j_1, \ldots, j_k)$ is an even permutation of $(i_1, \ldots, i_k)$, -1 if it is an odd permutation, and 0 if the two lists are not permutations of one another.

Exercise 9: Let α be an arbitrary element of $\Lambda^k(V^*)$. For each subset $I = (i_1, \ldots, i_k)$ written in increasing order, let $\alpha_I = \alpha(\vec{b}_{i_1}, \ldots, \vec{b}_{i_k})$. Show that $\alpha = \sum_I \alpha_I \phi^I$.

Exercise 10: Now let $\alpha_1, \ldots, \alpha_k$ be an arbitrary ordered list of covectors, and that $\vec{v}_1, \ldots, \vec{v}_k$ is an arbitrary ordered list of vectors. Show that $(\alpha_1 \wedge \cdots \wedge \alpha_k)(\vec{v}_1, \ldots, \vec{v}_k) = \det A$, where A is the $k \times k$ matrix whose i, j entry is $\alpha_i(\vec{v}_j)$.

3. Pullbacks

Suppose that $L : V \to W$ is a linear transformation, and that $\alpha \in \mathcal{T}^k(W^*)$. We then define the *pullback tensor* $L^*\alpha$ by

$$(L^*\alpha)(\vec{v}_1, \ldots, \vec{v}_k) = \alpha(L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k)). \tag{3}$$

This has some important properties. Pick bases $(\vec{b}_1, \ldots, \vec{b}_n)$ and $(\vec{d}_1, \ldots, \vec{d}_m)$ for V and W, respectively, and let $\{\phi^j\}$ and $\{\psi^j\}$ be the corresponding dual bases for V^* and W^*. Let A be the matrix of the linear transformation L relative to the two bases. That is

$$L(\vec{v})^j = \sum_j A_{ji} v^i.$$

Exercise 11: Show that the matrix of $L^* : W^* \to V^*$, relative to the bases $\{\psi^j\}$ and $\{\phi^j\}$, is A^T. [Hint: to figure out the components of a covector, act on a basis vector]
Exercise 12: If α is a k-tensor, and if $I = \{i_1, \ldots, i_k\}$, show that

$$(L^*\alpha)_I = \sum_{j_1, \ldots, j_k} A_{j_1, i_1} A_{j_2, i_2} \cdots A_{j_k, i_k} \alpha_{(j_1, \ldots, j_k)}.$$

Exercise 13: Suppose that α is alternating. Show that $L^*\alpha$ is alternating. That is, L^* restricted to $\Lambda^k(W^*)$ gives a map to $\Lambda^k(V^*)$.
Exercise 14: If α and β are alternating tensors on W, show that $L^*(\alpha\wedge\beta) = (L^*\alpha)\wedge(L^*\beta)$.

4. Cotangent bundles and forms

We're finally ready to define forms on manifolds. Let X be a k-manifold. An ℓ-dimensional *vector bundle* over X is a manifold E together with a surjection $\pi : E \to X$ such that

(1) The preimage $\pi^{-1}(p)$ of any point $p \in X$ is an n-dimensional real vector space. This vector space is called the *fiber* over p.
(2) For every point $p \in X$ there is a neighborhood U and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$, such that for each $x \in U$, ϕ_U restricted to $\pi^{-1}(x)$ is a linear isomorphism from $\pi^{-1}(x)$ to $x \times \mathbb{R}^n$ (where we think of $x \times \mathbb{R}^n$ as the vector space $\mathbb{R}^n$ with an additional label x.)

In practice, the isomorphism $\pi^{-1}(U) \to U \times \mathbb{R}^n$ is usually accomplished by defining a basis $(\vec{v}_1(x), \ldots, \vec{v}_n(x))$ for the fiber over x, such that each $\vec{v}_i$ is a smooth map from U to $\pi^{-1}(U)$.

Here are some examples of bundles:

- The tangent bundle $T(X)$. In this case $n = k$. If ψ is a local parametrization around a point p, then $d\psi$ applied to $e_1, \ldots, e_n$ give a basis for T_pX.
- The trivial bundle $X \times V$, where V is any n-dimensional vector space. Here we can pick a constant basis for V.
- The normal bundle of X in Y (where X is a submanifold of Y).
- The *cotangent bundle* whose fiber over x is the dual space of $T_x(X)$. This is often denoted $T_x^*(X)$, and the entire bundle is denoted $T^*(X)$. Given a smoothly varying basis for $T_x(X)$, we can take the dual basis for $T_x^*(X)$.
- The k-th tensor power of $T^*(X)$, which we denote $\mathcal{T}^k(T^*(X))$, i.e. the vector bundle whose fiber over x is $\mathcal{T}^k(T_x^*(X))$.
- The alternating k-tensors in $\mathcal{T}^k(T^*(X))$, which we denote $\Lambda^k(T^*(X))$.

Some key definitions:

- A *section* of a vector bundle $E \to X$ is a smooth map $s : X \to E$ such that $\pi \circ s$ is the identity on X. In other words, such that $s(x)$ is an element of the fiber over x for every x.
- A *differential form* of degree k is a section of $\Lambda^k(T^*(X))$. The (infinite-dimensional) space of k-forms on X is denoted $\Omega^k(X)$.
- If $f : X \to \mathbb{R}$ is a function, then $df_x : T_x(X) \to T_{f(x)}(\mathbb{R}) = \mathbb{R}$ is a covector at X. Thus every function f defines a 1-form df.

- If $f : X \to Y$ is a smooth map of manifolds, then df_x is a linear map $T_x(X) \to T_{f(x)}(Y)$, and so induces a *pullback map* $f^* : \Lambda^k(T^*_{f(x)}(Y)) \to \Lambda^k(T^*_x(X))$, and hence a linear map (also denoted f^*) from $\Omega^k(Y)$ to $\Omega^k(X)$.

Exercise 15: If $f : X \to Y$ and $g : Y \to Z$ are smooth maps of manifolds, then $g \circ f$ is a smooth map $X \to Z$. Show that $(g \circ f)^* = f^* \circ g^*$.

5. Reconciliation

We have developed two different sets of definitions for forms, pullbacks, and the d operator. Our task in this section is to see how they're really saying the same thing.

Old definitions:

- A differential form on $\mathbb{R}^n$ is a formal sum $\sum \alpha_I(x) dx^I$, where $\alpha_I(x)$ is an ordinary function and dx^I is a product $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ of meaningless symbols that anti-commute.
- The exterior derivative is $d\alpha = \sum_{I,j} (\partial_j \alpha_I(x)) dx^j \wedge dx^I$.
- If $g : \mathbb{R}^n \to \mathbb{R}^m$, then the pullback operator g is designed to pull back functions, commute with d, and respect wedge products: If $\alpha = \sum \alpha_I(y) dy^I$, then

$$g^*(\alpha)(x) = \sum_I \alpha_I(g(x)) dg^{i_1} \wedge \cdots \wedge dg^{i_k}.$$

- Forms on n-manifolds are defined via forms on $\mathbb{R}^n$ and local coordinates and have no intrinsic meaning.

New definitions:

- A differential form on $\mathbb{R}^n$ is a section of $\Lambda^k(T^*(\mathbb{R}^n))$. Its value at each point x is an alternating tensor that takes k tangent vectors at that point as inputs and outputs a number.
- The exterior derivative on functions is defined as the usual derivative map $df : T(X) \to \mathbb{R}$. We have not yet defined it for higher-order forms.
- If $g : X \to Y$, then the pullback map $\Omega^k(Y) \to \Omega^k(X)$ is induced by the derivative map $dg : T(X) \to T(Y)$.
- Forms on manifolds do not require a separate definition from forms on $\mathbb{R}^n$, since tangent spaces, dual spaces, and tensors on tangent spaces are already well-defined.

Our strategy for reconciling these two sets of definitions is:

(1) Show that forms on $\mathbb{R}^n$ are the same in both definitions.
(2) Extend the new definition of d to cover all forms, and show that it agrees with the old definition on Euclidean spaces.

(3) Show that the new definition of pullback, restricted to Euclidean spaces, satisfies the same axioms as the old definition, and thus gives the same operation on maps between Euclidean spaces, and in particular for change-of-coordinate maps.
(4) Show that the functional relations that were assumed when we extended the old definitions to manifolds are already satisfied by the new definitions.
(5) Conclude that the new definitions give a concrete realization of the old definitions.

On $\mathbb{R}^n$, the standard basis for the tangent space is $\{\vec{e}_1, \ldots, \vec{e}_n\}$. Since $\partial x^j/\partial x^i = \delta_i^j$, dx^j maps $\vec{e}_j$ to 1 and maps all other $\vec{e}_i$'s to zero. Thus the covectors $d_x x^1, \ldots, d_x x^n$ (meaning the derivatives of the functions $x^1, \ldots, x^n$ at the point x) form a basis for $T_x^*(\mathbb{R}^n)$ that is dual to $\{\vec{e}_1, \ldots, \vec{e}_n\}$. **In other words, $\phi^i = dx^i$!!** The meaningless symbols dx^i of the old definition are nothing more (or less) than the dual basis of the new definition. A new-style form is a linear combination $\sum_I \alpha_I \phi^I$ and an old-style form was a linear combination $\sum_I \alpha_I dx^I$, so the two definitions are exactly the same on $\mathbb{R}^n$. This completes step 1.

Next we want to extend the (new) definition of d to cover arbitrary forms. We would like it to satisfy $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$ and $d^2 = 0$, and that is enough.

$$\begin{aligned} d(dx^i) = d^2(x^i) &= 0 \\ d(dx^i \wedge dx^j) = d(dx^i) \wedge dx^j - dx^i \wedge d(dx^j) &= 0, \text{and similarly} \\ d(dx^I) &= 0 \text{ by induction on the degree of } I. \end{aligned} \tag{4}$$

This then forces us to take

$$\begin{aligned} d(\sum_I \alpha_I dx^I) &= \sum_I (d\alpha_I) \wedge dx^I + \alpha_I d(dx^I) \\ &= \sum_I (d\alpha_I) dx^I \\ &= \sum_{I,j} (\partial_j \alpha_I) dx^j \wedge dx^I, \end{aligned} \tag{5}$$

which is exactly the same formula as before. Note that this construction also works to define d uniquely on manifolds, as long as we can find functions f^i on a neighborhood of a point p such that the df^i's span $T_p^*(X)$. But such functions are always available via the local parametrization. If $\psi : U \to X$ is a local parametrization, then we can just pick f^i to be the i-th entry of ψ^{-1}. That is $f^i = x^i \circ \psi^{-1}$. This gives a formula for d on X that is equivalent to "convert to $\mathbb{R}^n$ using ψ, compute d in $\mathbb{R}^n$, and then convert back", which was our old definition of d on a manifold.

We now check that the definitions of pullback are the same. Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Under the new definition, $g^*(dy^i)(\vec{v}) = dy^i(dg(\vec{v}))$, which is the ith entry of $dg(\vec{v})$, where we are using coordinates $\{y^i\}$ on $\mathbb{R}^m$. But that is the same as $dg^i(\vec{v})$, so $g^*(dy^i) = dg^i$. Since the pullback of a function f is just the composition $f \circ g$, and since $g^*(\alpha \wedge \beta) = (g^*(\alpha)) \wedge (g^*(\beta))$ (see the last exercise in the "pullbacks" section), we must have

$$g^*(\sum_I \alpha_I dy^I)(x) = \sum_I \alpha_I(g(x)) dg^{i_1} \wedge \cdots \wedge dg^{i_k},$$

exactly as before. This also shows that $g^*(d\alpha) = d(g^*\alpha)$, since that identity is a consequence of the formula for g^*.

Next we consider forms on manifolds. Let X be an n-manifold, let $\psi : U \to X$ be a parametrization, where U is an open set in $\mathbb{R}^n$. Suppose that $a \in U$, and let $p = \psi(a)$. The standard bases for $T_a(\mathbb{R}^n)$ and $T_a^*(\mathbb{R}^n)$ are $\{\vec{e}_1, \ldots, \vec{e}_n\}$ and $\{dx^1, \ldots, dx^n\}$. Let $\vec{b}_i = dg_0(\vec{e}_i)$. The vectors $\{\vec{b}_i\}$ form a basis for $T_p(X)$. Let $\{\phi^j\}$ be the dual basis. But then

$$\begin{aligned} \psi^*(\phi^j)(\vec{e}_i) &= \phi^j(dg_a(\vec{e}_i)) \\ &= \phi^j(\vec{b}_i) \\ &= \delta_i^j \\ &= dx^j(\vec{e}_i), \text{ so} \\ \psi^*(\phi^j) &= dx^j. \end{aligned} \tag{6}$$

Under the old definition, forms on X were abstract objects that corresponded, via pullback, to forms on U, such that changes of coordinates followed the rules for pullbacks of maps $\mathbb{R}^n \to \mathbb{R}^n$. Under the new definition, ψ^* *automatically* pulls a basis for $T_p^*(X)$ to a basis for $T_a^*(\mathbb{R}^n)$, and this extends to an isomorphism between forms on a neighborhood of p and forms on a neighborhood of a. Furthermore, if $\psi_{1,2}$ are two different parametrizations of the same neighborhood of p, and if $\psi_1 = \psi_2 \circ g_{12}$ (so that g_{12} maps the ψ_1 coordinates to the ψ_2 coordinates), then we automatically have $\psi_1^* = g_{12}^* \circ \psi_2^*$, thanks to Exercise 15.

Bottom line: It is perfectly legal to do forms the old way, treating the dx's as meaningless symbols that follow certain axioms, and treating forms on manifolds purely via how they appear in various coordinate systems. However, sections of bundles of alternating tensors on $T(X)$ give an intrinsic realization of the exact same algebra. The new definitions allow us to talk about what differential forms actually *are*, and to develop a cleaner intuition on how forms behave. In particular, they give a very simple explanation of what integration over manifolds really means.

That's the subject of the next set of notes.