NOTES ON DIFFERENTIAL FORMS. PART 5: DE RHAM COHOMOLOGY

1. Closed and exact forms

Let X be a n-manifold (not necessarily oriented), and let α be a k-form on X. We say that α is *closed* if $d\alpha = 0$ and say that α is *exact* if $\alpha = d\beta$ for some $(k-1)$-form β. (When $k = 0$, the 0 form is also considered exact.) Note that

- Every exact form is closed, since $d(d\beta) = d^2\beta = 0$.
- A 0-form is closed if and only if it is locally constant, i.e. constant on each connected component of X.
- Every n-form is closed, since then $d\alpha$ would be an $(n+1)$-form on an n-dimensional manifold, and there are no nonzero $(n+1)$-forms.

Since the exact k-forms are a subspace of the closed k-forms, we can defined the quotient space

$$H^k_{dR}(X) = \frac{\text{Closed } k\text{-forms on } X}{\text{Exact } k\text{-forms on } X}.$$

This quotient space is called the kth *de Rham cohomology of* X. Since this is the only kind of cohomology we're going to discuss in these notes, I'll henceforth omit the prefix "de Rham" and the subscript $_{dR}$. If α is a closed form, we write $[\alpha]$ to denote the class of α in H^k, and say that the *form* α represents the *cohomology class* $[\alpha]$.

The wedge product of forms extends to a product operation $H^k(X) \times H^\ell(X) \to H^{k+\ell}(X)$. If α and β are closed, then

$$\begin{aligned} d(\alpha \wedge \beta) &= (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta \\ &= 0 \wedge \beta \pm \alpha \wedge 0 = 0, \end{aligned}$$

so $\alpha \wedge \beta$ is closed. Thus $\alpha \wedge \beta$ represents a class in $H^{k+\ell}$, and we define

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta].$$

We must check that this is well-defined. I'm going to spell this out in gory detail as an example of computations to come.

Suppose that $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$. We must show that $[\alpha' \wedge \beta'] = [\alpha \wedge \beta]$. However $[\alpha'] = [\alpha]$ means that α' and α differ by an exact form, and similarly for β' and β:

$$\alpha' = \alpha + d\mu$$

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$$\beta' = \beta + d\nu$$

But then

$$\begin{aligned} \alpha' \wedge \beta' &= (\alpha + d\mu) \wedge (\beta + d\nu) \\ &= \alpha \wedge \beta + (d\mu) \wedge \beta + \alpha \wedge d\nu + d\mu \wedge d\nu \\ &= \alpha \wedge \beta + d(\mu \wedge \beta) + (-1)^k d(\alpha \wedge \nu) + d(\mu \wedge d\nu) \\ &= \alpha \wedge \beta + \text{exact forms}, \end{aligned}$$

where we have used the fact that $d(\mu \wedge \beta) = d\mu \wedge \beta + (-1)^{k-1} \mu \wedge d\beta = d\mu \wedge \beta$, and similar expansions for the other terms. That is,

$$\begin{aligned} (\text{Exact}) \wedge (\text{Closed}) &= (\text{Exact}) \\ (\text{Closed}) \wedge (\text{Exact}) &= (\text{Exact}) \\ (\text{Exact}) \wedge (\text{Exact}) &= (\text{Exact}) \end{aligned}$$

Thus $\alpha' \wedge \beta'$ and $\alpha \wedge \beta$ represent the same class in cohomology. Since $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$, it also follows immediately that $[\beta] \wedge [\alpha] = (-1)^{k\ell}[\alpha] \wedge [\beta]$.

We close this section with a few examples.

- If X is a point, then $H^0(X) = \Omega^0(X) = \mathbb{R}$, and $H^k(X) = 0$ for all $k \neq 0$, since there are no nonzero forms in dimension greater than 1.
- If $X = \mathbb{R}$, then $H^0(X) = \mathbb{R}$, since the closed 0-forms are the constant functions, of which only the 0 function is exact. All 1-forms are both closed and exact. If $\alpha = \alpha(x)dx$ is a 1-form, then $\alpha = df$, where $f(x) = \int_0^x \alpha(s)ds$ is the indefinite integral of $\alpha(x)$.
- If X is *any* connected manifold, then $H^0(X) = \mathbb{R}$.
- If $X = S^1$ (say, embedded in $\mathbb{R}^2$), then $H^1(X) = \mathbb{R}$, and the isomorphism is obtained by integration: $[\alpha] \to \int_{S^1} \alpha$. If the form α is exact, then $\int_{S^1} \alpha = 0$. Conversely, if $\int_{S^1} \alpha = 0$, then $f(x) = \int_a^x \alpha$ (for an arbitrary fixed starting point a) is well-defined and $\alpha = df$.

2. Pullbacks in Cohomology

Suppose that $f : X \to Y$ and that α is a closed form on Y, representing a class in $H^k(Y)$. Then $f^*\alpha$ is also closed, since

$$d(f^*\alpha) = f^*(d\alpha) = f^*(0) = 0,$$

so $f^*\alpha$ represents a class in $H^k(X)$. If α' also represents $[\alpha] \in H^k(Y)$, then we must have $\alpha' = \alpha + d\mu$, so

$$f^*(\alpha') = f^*\alpha + f^*(d\mu) = f^*\alpha + d(f^*\mu)$$

represents the same class in $H^k(X)$ as $f^*\alpha$ does. We can therefore define a map

$$f^\sharp : H^k(Y) \to H^k(X), \qquad f^\sharp[\alpha] = [f^*\alpha].$$

We are using notation to distinguish between the pullback map f^* on *forms* and the pullback map $f^\sharp$ on *cohomology*. Guillemin and Pollack also follow this convention. However, most authors use f^* to denote both maps, hoping that it is clear from context whether we are talking about forms or about the classes they represent. (Still others use $f^\sharp$ for the map on forms and f^* for the map on cohomology. Go figure.)

Note that $f^\sharp$ is a *contravariant functor*, which is a fancy way of saying that it reverses the direction of arrows. If $f : X \to Y$, then $f^\sharp : H^k(X) \leftarrow H^k(Y)$. If $f : X \to Y$ and $g : Y \to Z$, then $g^\sharp : H^k(Z) \to H^k(Y)$ and $f^\sharp : H^k(Y) \to H^k(X)$. Since $(g \circ f)^* = f^* \circ g^*$, it follows that $(g \circ f)^\sharp = f^\sharp \circ g^\sharp$.

We will have more to say about pullbacks in cohomology after we have established some more machinery.

3. Integration over a fiber and the Poincare Lemma

Theorem 3.1 (Integration over a fiber). *Let X be any manifold. Let the* zero section *$s_0 : X \to \mathbb{R} \times X$ be given by $s_0(x) = (0, x)$, and let the* projection *$\pi : \mathbb{R} \times X \to X$ be given by $\pi(t, x) = x$. Then $s_0^\sharp : H^k(\mathbb{R} \times X) \to H^k(X)$ and $\pi^\sharp : H^k(X) \to H^k(\mathbb{R} \times X)$ are isomorphisms and are inverses of each other.*

Proof. Since $\pi \circ s_0$ is the identity on X, $s_0^\sharp \circ \pi^\sharp$ is the identity on $H^k(X)$. We must show that $\pi^\sharp \circ s_0^\sharp$ is the identity on $H^k(\mathbb{R} \times X)$. We do this by constructing a map $P : \Omega^k(\mathbb{R} \times X) \to \Omega^{k-1}(\mathbb{R} \times X)$, called a *homotopy operator,* such that for any k form α on $\mathbb{R} \times X$,

$$(1 - \pi^* \circ s^*) = d(P(\alpha)) + P(d\alpha).$$

If α is closed, this implies that α and $\pi^*(s_0^*\alpha)$ differ by the exact form $d(P(\alpha))$ and so represent the same class in cohomology, and hence that $\pi^\sharp \circ s_0^\sharp[\alpha] = [\alpha]$. Since this is true for all α, $\pi^\sharp \circ s_0^\sharp$ is the identity.

Every k-form on Y can be uniquely written as a product

$$\alpha(t, x) = dt \wedge \beta(t, x) + \gamma(t, x),$$

where β and γ have no dt factors. The $(k-1)$-form β can be written as a sum:

$$\beta(t, x) = \sum_J \beta_J(t, x) dx^J,$$

where $\beta_J(t,x)$ is an ordinary function, and we likewise write

$$\gamma(t,x) = \sum_I \gamma_I(t,x) dx^I.$$

We define

$$P(\alpha)(t,x) = \sum_J (\int_0^t \beta_J(s,x)ds)dx^J.$$

$P(\alpha)$ is called the *integral along the fiber* of α. Note that $s_0^*\alpha$, evaluated at x, is $\sum_I \gamma(0,x)dx^I$, and that

$$(1-\pi^* s_0^*)\alpha(t,x) = dt \wedge \beta(t,x) + \sum_I (\gamma(t,x) - \gamma(t,0))dx^I.$$

Now we compute $dP(\alpha)$ and $P(d\alpha)$. Since

$$d\alpha(t,x) = -dt \wedge \sum_{j,J} (\partial_j \beta_J(t,x))dx^j \wedge dx^J + \sum_I \partial_t \gamma_I(t,x) dt \wedge dx^I + \sum_{I,j} \partial_j \gamma_I(t,x) dx^j \wedge dx^I,$$

where j runs over the coordinates of X, we have

$$P(d\alpha)(t,x) = -\sum_{j,J} \int_0^t \Big(\partial_j \beta_J(s,x)ds\Big) dx^j \wedge dx^J + \sum_I \Big(\gamma_I(t,x) - \gamma_I(0,x)\Big) dx^I,$$

where we have used $\int_0^t \partial_s \gamma_I(s,x)ds = \gamma_I(t,x) - \gamma_I(0,x)$. Meanwhile,

$$d(P(\alpha)) = \sum_{j,J} \left(\int_0^t \partial_j \beta_J(s,x)ds \right) dx^j \wedge dx^J + \sum_J \beta_J(t,x) dt \wedge dx^J,$$

so

$$(dP + Pd)\alpha(t,x) = \sum_I \Big(\gamma_I(t,x) - \gamma_I(0,x)\Big) dx^I + dt \wedge \beta(t,x) = (1-\pi^* s_0^*)\alpha(t,x).$$

□

Exercise 1: In this proof, the operator P was defined relative to local coordinates on X. Show that this is in fact well-defined. That is, if we have two parametrizations ψ and ϕ, and we compute $P(\alpha)$ using the ϕ coordinates and then convert to the ψ coordinates, we get the same result as if we computed $P(\alpha)$ directly using the ψ coordinates.

An immediate corollary of this theorem is that $H^k(\mathbb{R}^n) = H^k(\mathbb{R}^{n-1}) = \cdots = H^k(\mathbb{R}^0)$. In particular,

Theorem 3.2 (Poincare Lemma). *On $\mathbb{R}^n$, or on any manifold diffeomorphic to $\mathbb{R}^n$, every closed form of degree 1 or higher is exact.*

Exercise 2: Show that a vector field $\vec{v}$ on $\mathbb{R}^3$ is the gradient of a function if and only if $\nabla \times \vec{v} = 0$ everywhere.

Exercise 3: Show that a vector field $\vec{v}$ on $\mathbb{R}^3$ can be written as a curl (i.e., $\vec{v} = \nabla \times \vec{w}$) if and only if $\nabla \cdot \vec{v} = 0$.

Exercise 4: Now consider the 3-dimensional torus $X = \mathbb{R}^3/\mathbb{Z}^3$. Construct a vector field $\vec{v}(x)$ whose curl is zero that is not a gradient (where we use the local isomorphism with $\mathbb{R}^3$ to define the curl and gradient). Construct a vector field $\vec{w}(x)$ whose divergence is zero that is not a curl.

In the integration-along-a-fiber theorem, we showed that $s_0^\sharp$ was the inverse of $\pi^\sharp$. However, we could have used the 1-section $s_1(x) = (1, x)$ instead of the 0-section and obtained the same result. (Just replace 0 with 1 everwhere that refers to a value of t). Thus

$$s_1^\sharp = (\pi^\sharp)^{-1} = s_0^\sharp.$$

This has important consequences for homotopies.

Theorem 3.3. *Homotopic maps induce the* same *map in cohomology. That is, if X and Y are manifolds and $f_{0,1} : X \to Y$ are smooth homotopic maps, then $f_1^\sharp = f_0^\sharp$.*

Proof. If f_0 and f_1 are homotopic, then we can find a smooth map $F : \mathbb{R} \times X \to Y$ such that $F(t, x) = f_0(x)$ for $t \leq 0$ and $F(t, x) = f_1(x)$ for $t \geq 1$. But then $f_1 = F \circ s_1$ and $f_0 = F \circ s_0$. Thus

$$f_1^\sharp = s_1^\sharp \circ F^\sharp = s_0^\sharp \circ F^\sharp = (F \circ s_0)^\sharp = f_0^\sharp.$$

□

Exercise 5: Recall that if A is a submanifold of X, then a *retraction* $r : X \to A$ (sometimes just called a *retract*) is a smooth map such that $r(a) = a$ for all $a \in A$. If such a map exists, we say that A is a *retract* of X. Suppose that $r : X \to A$ is such a retraction, and that i_A be the inclusion of A in X. Show that $r^\sharp : H^k(A) \to H^k(X)$ is surjective and $i_A^\sharp : H^k(X) \to H^k(A)$ is injective in every degree k. [We will soon see that $H^k(S^k) = \mathbb{R}$. This exercise, combined with the Poincare Lemma, will then provide another proof that there are no retractions from the unit ball in $\mathbb{R}^n$ to the unit sphere.]

Exercise 6: Recall that a *deformation retraction* is a retraction $r : X \to A$ such that $i_A \circ r$ is homotopic to the identity on X, in which case we say that A is a *deformation retract* of X. Suppose that A is a deformation retract of X. Show that $H^k(X)$ and $H^k(A)$ are isomorphic. [This provides another proof of the Poincare Lemma, insofar as $\mathbb{R}^n$ deformation retracts to a point.]

4. Mayer-Vietoris Sequences 1: Statement

Suppose that a manifold X can be written as the union of two open submanifolds, U and V. The Mayer-Vietoris Sequence is a technique for computing the cohomology of X from the cohomologies of U, V and $U \cap V$. This has direct practical importance, in that it allows us to compute things like $H^k(S^n)$ and many other simple examples. It also allows us to prove many properties of compact manifolds by induction on the number of open sets in a "good cover" (defined below). Among the things that can be proved with this technique (of which we will only prove a subset) are:

(1) $H^k(S^n) = \mathbb{R}$ if $k = 0$ or $k = n$ and is trivial otherwise.
(2) If X is compact, then $H^k(X)$ is finite-dimensional. This is hardly obvious, since $H^k(X)$ is the quotient of the infinite-dimensional vector space of closed k-forms by another infinite-dimensional space of exact k-forms. But as long as X is compact, the quotient is finite-dimensional.
(3) If X is a compact, oriented n-manifold, then $H^n(X) = \mathbb{R}$.
(4) If X is a compact, oriented n-manifold, then $H^k(X)$ is isomorphic to $H^{n-k}(X)$. (More precisely to the dual of $H^{n-k}(X)$, but every finite-dimensional vector space is isomorphic to its own dual.) This is called Poincare duality.
(5) If X is any compact manifold, orientable or not, then $H^k(X)$ is isomorphic to $Hom(H_k(X), \mathbb{R})$, where $H_k(X)$ is the k-th homology group of X.
(6) A formula for $H^k(X \times Y)$ in terms of the cohomologies of X and Y.

Suppose we have a sequence

$$V^1 \xrightarrow{L_1} V^2 \xrightarrow{L_2} V^3 \xrightarrow{L_3} \cdots$$

where each V^i is a vector space and each $L_i : V^i \to V^{i+1}$ is a linear transformation. We say that this sequence is *exact* if the kernel of each L_i equals the image of the previous L_{i-1}. In particular,

$$0 \to V \xrightarrow{L} W \to 0$$

is exact if and only if L is an isomorphism, since the kernel of L has to equal the image of 0, and the image of L has to equal the kernel of the 0 map on W.

Exercise 7: A *short exact sequence* involves three spaces and two maps:

$$0 \to U \xrightarrow{i} V \xrightarrow{j} W \to 0$$

Show that if this sequence is exact, there must be an isomorphism $h : V \to U \oplus W$, with $h \circ i(u) = (u, 0)$ and $j \circ h^{-1}(u, w) = w$.

Exact sequences can be defined for homeomorphisms between arbitrary Abelian groups, and not just vector spaces, but are much simpler when applied to vector spaces. In particular, the analogue of the previous exercise is *false* for groups. (E.g. one can define a short exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ even though $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

Suppose that $X = U \cup V$, where U and V are open submanifolds of X. There are natural inclusion maps i_U and i_V of U and V into X, and these induce maps i_U^* and i_V^* from $\Omega^k(X)$ to $\Omega^k(U)$ and $\Omega^k(V)$. Note that $i_U^*(\alpha)$ is just the restriction of α to U, while $i_V^*(\alpha)$ is the restriction of α to V. Likewise, there are inclusions ρ_U and ρ_V of $U \cap V$ in U and V, respectively, and associated restrictions ρ_U^* and ρ_V^* from $\Omega^k(U)$ and $\Omega^k(V)$ to $\Omega^k(U \cap V)$. Together, these form a sequence:

$$0 \to \Omega^k(X) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \to 0, \tag{1}$$

where the maps are defined as follows. If $\alpha \in \Omega^k(X)$, $\beta \in \Omega^k(U)$ and $\gamma \in \Omega^k(V)$, then

$$\begin{aligned} i_k(\alpha) &= (i_U^*\alpha, i_V^*\alpha) \\ j_k(\beta, \gamma) &= r_U^*\beta - r_V^*\gamma. \end{aligned}$$

Note that $d(i_k(\alpha)) = i_{k+1}(d\alpha)$ and that $d(j_k(\beta, \gamma)) = j_{k+1}(d\beta, d\gamma)$. That is, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k(X) & \xrightarrow{i_k} & \Omega^k(U) \oplus \Omega^k(V) & \xrightarrow{j_k} & \Omega^k(U \cup V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \\ 0 & \longrightarrow & \Omega^{k+1}(X) & \xrightarrow{i_{k+1}} & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) & \xrightarrow{j_{k+1}} & \Omega^{k+1}(U \cup V) & \longrightarrow & 0 \end{array}$$

commutes. Thus i_k and j_k send closed forms to closed forms and exact forms to exact forms, and induce maps

$$i_k^\sharp : H^k(X) \to H^k(U) \oplus H^k(V); \qquad j_k^\sharp : H^k(U) \oplus H^k(V) \to H^k(U \cap V).$$

Theorem 4.1 (Mayer-Vietoris)**.** *There exists a map $d_k^\sharp : H^k(U \cap V) \to H^{k+1}(X)$ such that the sequence*

$$\cdots H^k(X) \xrightarrow{i_k^\sharp} H^k(U) \oplus H^k(V) \xrightarrow{j_k^\sharp} H^k(U \cap V) \xrightarrow{d_k^\sharp} H^{k+1}(X) \xrightarrow{i_{k+1}^\sharp} H^{k+1}(U) \oplus H^{k+1}(V) \to \cdots$$

is exact.

The proof is a long slog, and warrants a section of its own. Then we will develop the uses of Mayer-Vietoris sequences.

5. Proof of Mayer-Vietoris

The proof has several big steps.

(1) We show that the sequence (1) of forms is actually exact.
(2) Using that exactness, and the fact that i and j commute with d, we then construct the map $d_k^\sharp$.

(3) Having constructed the maps, we show exactness at $H^k(U) \oplus H^k(V)$, i.e., that the image of $i_k^\sharp$ equals the kernel of $j_k^\sharp$.
(4) We show exactness at $H^k(U \cap V)$, i.e., that the image of $j_k^\sharp$ equals the kernel of $d_k^\sharp$.
(5) We show exactness at $H^{k+1}(X)$, i.e. that the kernel of $i_{k+1}^\sharp$ equals the image of $d_k^\sharp$.
(6) Every step but the first is formal, and applies just as well to *any* short exact sequence of (co)chain complexes. This construction in homological algebra is called the *snake lemma*, and may be familiar to some of you from algebraic topology. If so, you can skip ahead after step 2. If not, don't worry. We'll cover everything from scratch.

Step 1: Showing that

$$0 \to \Omega^k(X) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \to 0$$

amounts to showing that i_k is injective, that $Im(i_k) = Ker(j_k)$, and that j_k is surjective. The first two are easy. The subtlety is in showing that j_k is surjective.

Recall that i_U^*, i_V^*, r_U^* and r_V^* are all restriction maps. If $\alpha \in \Omega^k(X)$ and $i_k(\alpha) = 0$, then the restriction of α to U is zero, as is the restriction to V. But then α itself is the zero form on X. This shows that i_k is injective.

Likewise, for any $\alpha \in \Omega^k(X)$, $i_U^*(\alpha)$ and $i_V^*(\alpha)$ agree on $U \cap V$, so $r_U^* i_U^*(\alpha) = r_V^* i_V^*(\alpha)$, so $j_k(i_k(\alpha)) = 0$. Conversely, if $j_k(\beta, \gamma) = 0$, then $r_U^*(\beta) = r_V^*(\gamma)$, so we can stitch β and γ into a form α on X that equals β on U and equals γ on V (and equals both of them on $U \cap V$), so $(\beta, \gamma) \in Im(i_k)$.

Now suppose that $\mu \in \Omega^k(U \cap V)$ and that $\{\rho_U, \rho_V\}$ is a partition of unity of X relative to the open cover $\{U, V\}$. Since the function ρ_U is zero outside of U, the form $\rho_U \mu$ can be extended to a smooth form on V by declaring that $\rho_U \mu = 0$ on $V - U$. Note that $\rho_U \mu$ is **not** a form on U, since μ is not defined on the entire support of ρ_U. Rather, $\rho_U \mu$ is a form on V, since μ is defined at all points of V where $\rho_U \neq 0$. Likewise, $\rho_V \mu$ is a form on U. On $U \cap V$, we have $\mu = \rho_V \mu - (-\rho_U \mu)$. This means that $\mu = j_k(\rho_V \mu, -\rho_U \mu)$.

The remaining steps are best described in the language of homological algebra. A *cochain complex* A is a sequence of vectors spaces[1] $\{A^0, A^1, A^2, \ldots\}$ together with maps $d_k : A^k \to A^{k+1}$ such that $d_k \circ d_{k-1} = 0$. We also define $A^{-1} = A^{-2} = \cdots$ to be 0-dimensional vector spaces ("0") and $d_{-1} = d_{-2} = \cdots$ to be the zero map. The *k-th cohomology of the complex* is

$$H^k(A) = \frac{\text{kernel of } d_k}{\text{image of } d_{k-1}}.$$

[1]For (co)homology theories with integer coefficients one usually considers sequences of Abelian groups. For de Rham theory, vector spaces will do.

A *cochain map* $i : A \to B$ between complexes A and B is a family of maps $i_k : A^k \to B^k$ such that $d_k(i_k(\alpha)) = i_{k+1}(d_k\alpha)$ for all k and all $\alpha \in A^k$, i.e., such that the diagram

$$\begin{array}{ccc} A^k & \xrightarrow{i_k} & B^k \\ \Big\downarrow d_k^A & & \Big\downarrow d_k^B \\ A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} \end{array}$$

commutes for all k, where we have labeled the two differential maps d_k^A and d_k^B to emphasize that they are defined on different spaces.

Exercise 8: Show that a cochain map $i : A \to B$ induces maps in cohomology

$$i_k^\sharp : H^k(A) \to H^k(B); \qquad [\alpha] \mapsto [i_k\alpha].$$

If A, B and C are cochain complexes and $i : A \to B$ and $j : B \to C$ are cochain maps, then the sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is said to be a *short exact sequence* of cochain complexes if, for each k,

$$0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \to 0$$

is a short exact sequence of vector spaces.

So far, we have constructed a short exact sequence of cochain complexes with $A^k = \Omega^k(X)$, $B^k = \Omega^k(U) \oplus \Omega^k(V)$, and $C^k = \Omega^k(U \cap V)$. The theorem on Mayer-Vietoris sequences is then a special case of

Theorem 5.1 (Snake Lemma)**.** *Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of cochain complexes. Then there is a family of maps $d_k^\sharp : H^k(C) \to H^{k+1}(A)$ such that the sequence*

$$\cdots \to H^k(A) \xrightarrow{i_k^\sharp} H^k(B) \xrightarrow{j_k^\sharp} H^k(C) \xrightarrow{d_k^\sharp} H^{k+1}(A) \xrightarrow{i_{k+1}^\sharp} H^{k+1}(B) \to \cdots$$

is exact.

We will use the letters α, β, γ, with appropriate subscripts and other markers, to denote elements of A, B and C, respectively. For simplicity, we will write "d" for d_k^A, d_k^B, d_k^C, d_{k+1}^A, etc. Our first task is to define $d_k^\sharp[\gamma_k]$, where $[\gamma_k]$ is a class in $H^k(C)$.

Since j_k is surjective, we can find $\beta_k \in B^k$ such that $\gamma_k = j_k(\beta_k)$. Now,

$$j_{k+1}(d\beta_k) = d(j_k(\beta_k)) = d(\gamma_k) = 0,$$

since γ_k was closed. Since $d\beta_k$ is in the kernel of j_{k+1}, it must be in the image of i_{k+1}. Let α_{k+1} be such that $i_{k+1}(\alpha_{k+1}) = d\beta_k$. Furthermore, α_{k+1} is unique, since i_{k+1} is injective. We define

$$d_k^\sharp[\gamma_k] = [\alpha_{k+1}].$$

The construction of $[\alpha_{k+1}]$ from $[\gamma_k]$ is summarized in the following diagram:

$$\begin{array}{ccccc} & & \beta_k & \xrightarrow{j_k} & \gamma_k \\ & & \big\downarrow d & & \\ \alpha_{k+1} & \xrightarrow{i_{k+1}} & d\beta_k & & \end{array}$$

For this definition to be well-defined, we must show that

- α_{k+1} is closed. However,

$$i_{k+2}(d\alpha_{k+1}) = d(i_{k+1}\alpha_{k+1}) = d(d\beta_k) = 0.$$

  Since i_{k+2} is injective, $d\alpha_{k+1}$ must then be zero.
- The class $[\alpha_{k+1}]$ does not depend on which β_k we chose, so long as $j_k(\beta_k) = \alpha_k$. The argument in displayed in the diagram

$$\begin{array}{ccccc} (\alpha_k) & & \beta'_k = \beta_k + i_k(\alpha_k) & \xrightarrow{j_k} & \gamma_k \\ & & \big\downarrow d & & \\ \alpha'_{k+1} = \alpha_{k+1} + d\alpha_k & \xrightarrow{i_{k+1}} & d\beta'_k = d\beta_k + di_k\alpha_k & & \end{array}$$

  To see this, suppose that we pick a different $\beta' \in B^k$ with $j_k(\beta'_k) = \gamma_k = j_k(\beta_k)$. Then $j_k(\beta'_k - \beta_k) = 0$, so $\beta'_k - \beta_k$ must be in the image of i_k, so there exists $\alpha_k \in A^k$ such that $\beta'_k = \beta_k + i_k(\alpha_k)$. But then

$$d\beta' = d\beta + d(i_k(\alpha_k)) = d\beta + i_{k+1}(d\alpha_k) = i_{k+1}(\alpha_{k+1} + d\alpha_k),$$

  so $\alpha'_{k+1} = \alpha_{k+1} + d\alpha_k$. But then $[\alpha'_{k+1}] = [\alpha_{k+1}]$, as required.
- The class $[\alpha_{k+1}]$ does not depend on which cochain γ_k we use to represent the class $[\gamma_k]$.

$$\begin{array}{ccccc} & & \beta_{k-1} & \xrightarrow{j_{k-1}} & \gamma_{k-1} \\ & & & & \\ & & \beta_k + d\beta_{k-1} & \xrightarrow{j_k} & \gamma_k + d\gamma_{k-1} \\ & & \big\downarrow d & & \\ \alpha_{k+1} & \xrightarrow{i_{k+1}} & d\beta_k & & \end{array}$$

  Suppose $\gamma'_k = \gamma_k + d\gamma_{k-1}$ is another representative of the same class. Then there exists a β_{k-1} such that $\gamma_{k-1} = j_{k-1}\beta_{k-1}$. But then

$$j_k(\beta_k + d\beta_{k-1}) = j_k(\beta_k) + j_k(d(\beta_{k-1})) = \gamma_k + d(j_{k-1}\beta_{k-1}) = \gamma_k + d\gamma_{k-1} = \gamma'.$$

Thus we can take $\beta_k' = \beta_k + d\beta_{k-1}$. But then $d\beta_k' = d\beta_k$, and our cochain α_{k+1} is exactly the same as if we had worked with γ instead of γ'.

Before moving on to the rest of the proof of the Snake Lemma, let's stop and see how this works for Mayer-Vietoris.

(1) Start with a class in $H^k(U \cap V)$, represented by a closed form $\gamma_k \in \Omega^k(U \cap V)$.
(2) Pick $\beta_k = (\rho_V \gamma_k, -\rho_U \gamma_k)$, where (ρ_U, ρ_V) is a partition of unity.
(3) $d\beta_k = (d(\rho_V \gamma_k), -d(\rho_U \gamma_k))$. This is zero outside of $U \cap V$, since $\rho_V \gamma_k$ and $\rho_U \gamma_k$ were constructed to be zero outside $U \cap V$.
(4) Since $\rho_U = 1 - \rho_V$ where both are defined, $d\rho_U = -d\rho_V$. Since $d\gamma_k = 0$, we then have $-d(\rho_U \gamma_k) = d(\rho_V \gamma_k)$ on $U \cap V$. This means that the forms $d(\rho_V \gamma_k)$ on U and $-d(\rho_U \gamma_k)$ on V agree on $U \cap V$, and can be stitched together to define a closed form α_{k+1} on all of X. That is, $d_k^\sharp[\gamma_k]$ is represented by the closed form

$$d_k^*(\gamma_k) = \begin{cases} d(\rho_V \gamma_k) & \text{on } U \\ -d(\rho_U \gamma_k) & \text{on } V, \end{cases}$$

and the two definitions agree on $U \cap V$.

Returning to the Snake Lemma, we must show six inclusions:

- $Im(i_k^\sharp) \subset Ker(j_k^\sharp)$, i.e. that $j_k^\sharp i_k^\sharp[\alpha_k] = 0$ for any closed cochain α_k. This follows from

$$j_k^\sharp(i_k^\sharp[\alpha_k]) = j_k^\sharp[i_k \alpha_k] = [j_k(i_k(\alpha_k))] = 0,$$

  since $j_k \circ i_k = 0$.
- $Im(j_k^\sharp) \subset Ker(d_k^\sharp)$, i.e. that $d_k^\sharp \circ j_k^\sharp = 0$. If $[\beta_k]$ is a class in $H^k(B)$, then $j_k^\sharp[\beta_k] = [j_k \beta_k]$. To apply $d_k^\sharp$ to this, we must
  (a) find a cochain in B^k that maps to $j_k \beta_k$. Just take β_k itself!
  (b) Take d of this cochain. That gives 0, since β_k is closed.
  (c) Find an α_{k+1} that maps to this. This is $\alpha_{k+1} = 0$.
- $Im(d_k^\sharp) \subset Ker(i_{k+1}^\sharp)$. If α_{k+1} represents $d_k^\sharp[\gamma_k]$, then $i_{k+1}\alpha_{k+1} = d\beta_k$ is exact, so $i_{k+1}^\sharp[\alpha_k] = 0$.
- $Ker(j_k^\sharp) \subset Im(i_k^\sharp)$. If $[j_k \beta_k] = 0$, then $j_k \beta = d(\gamma_{k-1})$. Since j_{k-1} is surjective, we can find a β_{k-1} such that $j_{k-1}\beta_{k-1} = \gamma_{k-1}$. But then $j_k(d\beta_{k-1}) = d(j_{k-1}(\beta_{k-1})) = d\gamma_{k-1} = \gamma_k$. Since $j_k(\beta_k - d\beta_{k-1}) = 0$, we must have $\beta_k - d\beta_{k-1} = i_k(\alpha_k)$ for some $\alpha_k \in A^k$. Note that $i_{k+1} d\alpha_k = d(i_k \alpha_k) = d\beta_k - d(d\beta_{k-1}) = 0$, since β_k is closed. But i_{k+1} is injective, so $d\alpha_k$ must be zero, so α_k represents a class $[\alpha_k] \in H^k(A)$. But then $i_k^\sharp[\alpha_k] = [i_k(\alpha_k)] = [\beta_k + d\beta_{k-1}] = [\beta_k]$, so $[\beta_k]$ is in the image of $i_k^\sharp$.
- $Ker(d_k^\sharp) \subset Im(j_k^\sharp)$. Suppose that $d_k^\sharp[\gamma_k] = 0$. This means that the α_{k+1} constructed to satisfy $i_{k+1}\alpha_{k+1} = d\beta_k$, where $\gamma_k = j_k(\beta_k)$, must be exact. That is, $\alpha_{k+1} = d\alpha_k$ for

some $\alpha_k \in A^k$. But then

$$d\beta_k = i_{k+1}(\alpha_{k+1}) = i_{k+1}(d\alpha_k) = d(i_k(\alpha_k)).$$

Thus $\beta_k - i_k(\alpha_k)$ must be closed, and must represent a class in $H^k(B)$. But

$$j_k^\sharp[\beta_k - i_k(\alpha_k)] = [j_k\beta_k - j_k(i_k(\alpha_k))] = [j_k(\beta_k)] = [\gamma_k],$$

since $j_k \circ i_k = 0$. Thus $[\gamma_k]$ is in the image of $j_k^\sharp$.

- $Ker(i_{k+1}^\sharp) \subset Im(d_k^\sharp)$. Let α_{k+1} be a closed cochain in A^{k+1}, and suppose that $i_{k+1}^\sharp[\alpha_{k+1}] = 0$. This means that $i_{k+1}\alpha_{k+1}$ is exact, and we can find a $\beta_k \in B^k$ such that $i_{k+1}\alpha_{k+1} = d\beta_k$. Note that $dj_k\beta_k = j_{k+1}d\beta_k = j_{k+1}(i_{k+1}(\alpha_{k+1})) = 0$, so $j_k\beta_k$ is closed. But then $[j_k\beta_k] \in H^k(C)$ and $[\alpha_{k+1}] = d_k^\sharp[j_k\beta_k]$ is in the image of $d_k^\sharp$.

VERY IMPORTANT Exercise 9: Each of these arguments is called a "diagram chase", in that it involves using properties at one spot of a commutative diagram to derive properties at an adjacent spot. For each of these arguments, draw an appropriate diagram to illustrate what is going on. (In the history of the universe, nobody has fully understood the Snake Lemma without drawing out the diagrams himself or herself.)

6. Using Mayer-Vietoris

Our first application of Mayer-Vietoris will be to determine $H^k(S^n)$ for all k and n. We begin with the 1-sphere, whose cohomology we already computed using other methods. Here we'll compute it in detail using Mayer-Vietoris, as an example of how the machinery works.

Let S^1 be the unit circle embedded in $\mathbb{R}^2$. Let $U = \{(x,y) \in S^1 | y < 1/2\}$ and let $V = \{(x,y) \in S^1 | y > -1/2\}$. The open sets U and V are both diffeomorphic to $\mathbb{R}$, so $H^0(U) = H^0(V) = \mathbb{R}$ and $H^k(U) = H^k(V) = 0$ for $k > 0$. $U \cap V$ consists of two intervals, one with $x > 0$ and the other with $x < 0$. Each is diffeomorphic to the line, so $H^0(U \cap V) = \mathbb{R}^2$ and $H^k(U \cap V) = 0$ for $k > 0$. The Mayer-Vietoris sequence

$$0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(S^1) \to H^1(U) \oplus H^1(V) \to \cdots$$

simplifies to

$$0 \to H^0(S^1) \xrightarrow{i_0^\sharp} \mathbb{R}^2 \xrightarrow{j_0^\sharp} \mathbb{R}^2 \xrightarrow{d_0^\sharp} H^1(S^1) \xrightarrow{j_1^\sharp} 0$$

1. Since S^1 is connected, $H^0(S^1) = \mathbb{R}$.
2. Since $i_0^\sharp$ is injective, the image of $i_0^\sharp$ is 1-dimensional.
3. This makes the kernel of $j_0^\sharp$ 1-dimensional, so $j_0^\sharp$ has rank 1.
4. This makes the kernel of $d_0^\sharp$ 1-dimensional, so $d_0^\sharp$ has rank 1.
5. This make $H^1(S^1)$ 1-dimensional, and we conclude that $H^1(S^1) = H^0(S^1) = \mathbb{R}$ (and $H^k(S^1) = 0$ for $k > 1$ since S^1 is only 1-dimensional).

What's more, we can use Mayer-Vietoris to find a generator of $H^1(S^1)$. We just take an element of $H^0(U\cap V)$ that is not in the image of $j_0^\sharp$ and apply $d_0^\sharp$ to it. In fact, let's see what all the maps in the Mayer-Vietoris sequence are.

$H^0(S^1)$ and $H^0(U)$ and $H^0(V)$ are each generated by the constant function 1. Restricting 1 from X to U or from X to V gives a function that is 1 on U (or V), and restricting 1 from U to $U\cap V$, or from V to $U\cap V$, gives a function that is 1 on both components of $U\cap V$. Thus

$$i_0^\sharp(s) = (s,s) \qquad j_0^\sharp(s,t) = (s-t, s-t).$$

Now consider $(1,0)\in H^0(U\cap V)$. This is the function γ_0 that is 1 on one component of $U\cap V$ (say, the piece with $x>0$) and 0 on the other component. Now let ρ_U be a smooth function that is 1 for $y<-1/10$ and is 0 for $y>1/10$, and let $\rho_V = 1-\rho_U$. Then $\rho_V\gamma_0$ is a function on U that is

- Equal to ρ_V on the part of $U\cap V$ with $x>0$.
- Equal to 0 on the part of $U\cap V$ with $x<0$.
- Equal to 0 on the rest of U since there $\rho_V=0$.

Similarly, $-\rho_U\gamma$ is a function on V that is only nonzero on the part of $U\cap V$ where $x>0$. We then have α_1 is a 1-form that is

- Equal to $d\rho_V = -d\rho_U$ on the part of $U\cap V$ with $x>0$, and
- Equal to 0 everywhere else.

Since ρ_V increases from 0 to 1 as we move counterclockwise along this interval, $\int_{S^1}\alpha_1 = 1$. In fact, the support of α_1 is only a small part of $U\cap V$, and is included in the region where $x>0$ and $-1/10\le y\le 1/10$. This is called a "bump form", in analogy with bump functions.

Now we compute the cohomology of the n-sphere S^n. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and let U and V be the portions of that sphere with $x_{n+1}<1/2$ and with $x_{n+1}>-1/2$. Each of U and V is diffeomorphic to $\mathbb{R}^n$, so $H^k(U)=H^k(V)=\mathbb{R}$ for $k=0$ and $H^k(U)=H^k(V)=0$ otherwise. $U\cap V$ is a strip around the equator and is diffeomorphic to $\mathbb{R}\times S^{k-1}$, and so has the same cohomology as S^{n-1}. Since $H^k(U)=H^k(V)=0$ for $k>0$, the sequence

$$H^k(U)\oplus H^k(V)\to H^k(U\cap V)\to H^{k+1}(S^n)\to H^{k+1}(U)\oplus H^{k+1}(V)$$

is

$$0\to H^k(S^{n-1})\to H^{k+1}(S^n)\to 0,$$

so $H^{k+1}(S^n)$ is isomorphic to $H^k(S^{n-1})$. By induction on n, this shows that $H^k(S^n)=\mathbb{R}$ when $k=n$ and is 0 when $0<k\neq n$. Furthermore, the generator of $H^n(S^n)$ can be realized as a bump form, equal to $d\rho_V$ wedged with the generator of $H^{n-1}(S^{n-1})$, which in turn is a

bump 1-form wedged with the generator of $H^{n-1}(S^{n-2})$. Combining steps, this gives a bump n-form of total integral 1, localized near the point $(1, 0, 0, \ldots, 0)$.

Exercise 10: Let $T = S^1 \times S^1$ be the 2-torus. By dividing one of the S^1 factors circle two (overlapping) open sets, we can divide T into two cylinders U and V, such that $U \cap V$ is itself the disjoint union of two cylinders. Use this partition and the Mayer-Vietoris sequence to compute the cohomology of X. Warning: unlike with the circle, the dimensions of $H^k(U)$, $H^k(V)$ and $H^k(U \cap V)$ are not enough to solve this problem. You have to actually study what $i_k^\sharp$, $j_k^\sharp$ and/or $d_k^\sharp$ are doing. [Note: $H^1(\mathbb{R} \times S^1) = \mathbb{R}$, by integration over the fiber. This theorem also implies that the generator of $H^1(\mathbb{R} \times S^1)$ is just the pullback of a generator of $H^1(S^1)$ to $\mathbb{R} \times S^1$. You should be able to explicitly write generators for $H^1(U)$, $H^1(V)$ and $H^1(U \cap V)$ and see how the maps $i_1^\sharp$ and $j_1^\sharp$ behave.]

Exercise 11: Let K be a Klein bottle. Find open sets U and V such that $U \cup V = K$, such that U and V are cylinders, and such that $U \cap V$ is the disjoint union of two cylinders. In other words, the exact same data as with the torus T. The difference is in the ranks of some of the maps. Use Mayer-Vietoris to compute the cohomology of K.

7. Good covers and the Mayer-Vietoris argument

A set is *contractible* if it deformation retracts to a single point, in which case it has the same cohomology as a single point, namely $H^0 = \mathbb{R}$ and $H^k = 0$ for $k > 0$. In the context of n-manifolds, an open contractible set is something diffeomorphic to $\mathbb{R}^n$.

Exercise 12: Suppose that we are on a manifold with a Riemannian metric, so that there is well-defined notion of geodesics. Suppose furthermore that we are working on a region on which geodesics are unique: there is one and only one geodesic from a point p to a point q. An open submanifold A is called *convex* if, for any two points $p, q \in A$, the geodesic from p to q is entirely in A. Show that a convex submanifold is contractible.

Exercise 13: Show that the (non-empty) intersection of any collection of convex sets is convex, and hence contractible.

A *good cover* of a topological space is an open cover $\{U_i\}$ such that any finite intersection $U_{i_1} \cap U_{i_1} \cap \cdots \cap U_{i_k}$ is either empty or contractible.

Theorem 7.1. *Every compact manifold X admits a finite good cover.*

Proof. First suppose that X has a Riemannian metric. Then each point has a convex geodesic (open) neighborhood. Since the intersection of two (or more) convex sets is convex, these neighborhoods form a good cover. Since X is compact, there is a finite sub-cover, which is still good.

If X is embedded in $\mathbb{R}^N$, then the Riemannian structure comes from the embedding. The only question is how to get a Riemannian metric when X is an *abstract* manifold. To do this, partition X into coordinate patches. Use the Riemannian metric on $\mathbb{R}^n$ on each patch. Then stitch them together using partitions of unity. Since any positive linear combination of inner products still satisfies the axioms of an inner product, this gives a Riemannian metric for X. □

We next consider how big the cohomology of a manifold X can be. $H^k(X)$ is the quotient of two infinite-dimensional vector spaces. Can the quotient be infinite-dimensional?

If X is not compact, it certainly can. For example, consider the connected sum of an infinite sequence of tori. H^1 of such a space would be infinite-dimensional. However,

Theorem 7.2. *If X is a compact n-manifold, then each $H^k(X)$ is finite-dimensional.*

Proof. The proof is by induction on the number of sets in a good cover.

- If a manifold X admits a good cover with a single set U_1, then X is either contractible or empty, so $H^0(X) = \mathbb{R}$ or 0 and all other cohomology groups are trivial.
- Now suppose that all manifolds (compact or not) that admit open covers with at most m elements have finite-dimensional cohomology, and suppose that X admits a good cover $\{U_1, \ldots, U_{m+1}\}$. Let $U = U_1 \cup \cdots U_m$ and let $V = U_{m+1}$. But then $\{U_1 \cap U_{m+1}, U_2 \cap U_{m+1}, \cdots, U_m \cap U_{m+1}$ is a good cover for $U \cap V$ with m elements, so the cohomologies of U, V and $U \cap V$ are finite-dimensional.
- The Mayer-Vietoris sequence says that $H^{k-1}(U \cap V) \xrightarrow{d^\sharp_{k-1}} H^k(X) \xrightarrow{i^\sharp_k} H^k(U) \oplus H^k(V)$ is exact. However, $H^{k-1}(U \cap V)$ and $H^k(U) \oplus H^k(V)$ are finite dimensional, since $U \cap V$, U and V all admit good covers with at most m elements. Thus $H^k(X)$ must also be finite-dimensional.
- By induction, all manifolds with finite good covers have finite-dimensional cohomologies.
- Since all compact manifolds have finite good covers, all compact manifolds have finite-dimensional cohomologies.

□

This proof was an example of the *Mayer-Vietoris argument.* In general, we might want to prove that all spaces with finite good covers have a certain property P. Examples of such properties include finite-dimensional cohomology, Poincare duality (between de Rham cohomology and something called "compactly supported cohomology"), the Kunneth formula for cohomologies of product spaces, and the isomorphism between de Rham cohomology and singular cohomology with real coefficients. The steps of the argument are the same for all of these theorems:

(1) Show that every contractible set has property P.
(2) Using the Mayer-Vietoris sequence, show that if U, V, and $U \cap V$ have property P, then so does $U \cup V$.
(3) Proceeding by induction as above, showing that all manifolds with finite good covers have property P.
(4) Conclude that all compact manifolds have property P.

For lots of examples of the Mayer-Vietoris principle in action, see Bott and Tu's excellent book *Differential forms in algebraic topology.*