NOTES ON DIFFERENTIAL FORMS. PART 6: TOP COHOMOLOGY, POINCARE DUALITY, AND DEGREE

1. Compactly supported cohomology

Integration is a pairing between compactly supported forms and oriented manifolds. Given an oriented manifold X and a compactly supported n-form ω, we compute $\int_X \omega$. Of course, if X is compact, then *every* form on X is compactly supported. Also, if X is not compact and ω is not compactly supported, then we can often compute $\int_X \omega$ via limits. But at its core, integration is about compactly supported forms.

As such, it makes sense to analyze the cohomology of compactly supported forms. Let $\Omega^k_c(X)$ denote the vector space of compactly supported k-forms on the n-manifold X. Since d of a compactly supported form is compactly supported, we have a complex:

$$0 \to \Omega^1_c(X) \xrightarrow{d} \Omega^2_c(X) \xrightarrow{d} \Omega^3_c(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_c(X) \to 0,$$

and we define $H^k_c(X)$ to be the k-th cohomology of this complex. That is,

$$H^k_c(X) = \frac{\text{Closed, compactly supported } k\text{-forms on } X}{d(\text{Compactly supported } (k-1)\text{-forms on } X)}.$$

As with ordinary (de Rham) cohomology, we start by studying the cohomology of $\mathbb{R}^n$.

Theorem 1.1. *$H^k_c(\mathbb{R}^n) = \mathbb{R}$ if $k = n$ and 0 otherwise.*

Proof. I will treat the cases $n = 0$, $n = 1$ and $n = 2$ by hand, and then show how to get all larger values of n by induction.

If $n = 0$, then $\mathbb{R}^n$ is compact, and $H^0_c(\mathbb{R}^0) = H^0(\mathbb{R}^0) = \mathbb{R}$.

If $n = 1$, then H^0_c consists of compactly supported constant functions. But a constant function is only compactly supported if the constant is zero! Thus $H^0_c(\mathbb{R}) = 0$. As for H^1_c, suppose that α is a 1-form supported on the interval $[-R, R]$. Let $f(x) = \int_{-R}^x \alpha$. Then $\alpha = df$, and f is the only antiderivative of α that is zero for $x < -R$. Meanwhile, $f(x) = 0$ for $x > R$ if and only if $\int_{\mathbb{R}} \alpha = 0$, so α is d of a compactly supported function if and only if

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$\int_{\mathbb{R}} \alpha = 0$, and

$$H^1_c(\mathbb{R}) = \frac{\text{All compactly supported 1-forms on } \mathbb{R}}{\text{Compactly supported 1-forms on } \mathbb{R} \text{ with integral zero.}} = \mathbb{R}.$$

If $n = 2$, then $H^0_c(\mathbb{R}^2)$ is trivial, since the only compactly supported constant function is zero. We also have that $H^1_c(\mathbb{R}^2) = 0$, since if α is a closed 1-form supported on a (closed subset of a) ball of radius R around the origin, and if p is a point outside that ball, then $\alpha = df$, where $f(x) = \int_p^x \alpha$. (This integral doesn't depend on the path chosen from p to x because α is closed and $\mathbb{R}^2$ is simply connected.) The function f is supported on the ball of radius R, since if $|x| > R$, then there is a path from p to x that avoids the support of α altogether.

The tricky thing is showing that a compactly supported 2-form β on $\mathbb{R}^2$ is d of a compactly supported 1-form if and only if $\int_{\mathbb{R}^2} \beta = 0$.

The "only if" part is just Stokes' Theorem. If $\beta = d\gamma$, with γ compactly supported, then $\int_{\mathbb{R}^2} \beta = \int_{\mathbb{R}^2} d\gamma = \int_{\partial \mathbb{R}^2} \gamma = 0$.

To prove the other implication, suppose that $\beta = b(x, y)dx \wedge dy$ is a compactly supported 2-form of total integral zero, say supported on a closed subset of the square $[-R, R] \times [-R, R]$ for some $R > 1$. We define a number of useful functions and forms as follows:

- Let $f(s)$ be a smooth function of $\mathbb{R}$ with $f(s) = 1$ for $s \geq 1$ and $f(s) = 0$ for $s \leq 0$. Then $df = f'(s)ds$ is a bump 1-form, supported on $[0, 1]$, of total integral 1.
- Let $B(x) = \int_{-R}^R b(x, y)dy$. Note that B is compactly supported and that $\int_{\mathbb{R}} B dx = 0$. By our 1-dimensional analysis, there is a compactly supported function $G(x)$ such that $dG = B(x)dx$.
- Let $\tilde{\beta} = B(x)f'(y)dx \wedge dy$. Note that this is d of $G(x)f'(y)dy$, which in turn is a compactly supported 1-form.
- Now let $C(x, y) = \int_{-R}^y (b(x, s) - B(x)f'(s))ds$. This is compactly supported since $\int_{-R}^R (b(x, s) - f'(s)B(x))ds = 0$.
- $d(-C(x, y)dx) = \frac{\partial C}{\partial y} dx \wedge dy = (b(x, y) - B(x)f'(y))dx \wedge dy = \beta - \tilde{\beta}$.
- Since both $\tilde{\beta}$ and $\beta - \tilde{\beta}$ can be written as d of a compactly supported 1-form, so can β.

To go beyond $n = 2$, we need a variant on the integration-over-a-fiber argument that we previously used to get the Poincare Lemma. We want to compare the cohomologies of X

and $X \times \mathbb{R}$.[1] We will construct maps

$$i_k : \Omega^k_c(X) \to \Omega^{k+1}_c(X \times \mathbb{R}); \qquad j_{k+1} : \Omega^{k+1}_c(X \times \mathbb{R}) \to \Omega^k_c(X)$$

and a *homotopy operator* $P_{k+1} : \Omega^{k+1}_c(X \times \mathbb{R}) \to \Omega^k_c(X \times \mathbb{R})$ such that

$$\begin{aligned} d \circ i &= i \circ d \\ d \circ j &= j \circ d \\ j \circ i &= 1 \\ (1 - i \circ j) &= \pm dP \pm Pd, \end{aligned} \tag{1}$$

where we have suppressed the subscripts on i_k, j_k, d_k, and P_k and the identity map 1, and where the signs in the last equation may depend on k. The first line implies that i induces a map $i^\sharp : H^k_c(X) \to H^{k+1}_c(X \times \mathbb{R})$, and the second that j induces a map $j^\sharp : H^{k+1}_c(X \times \mathbb{R}) \to H^k_c(X)$. The third line implies that $j^\sharp \circ i^\sharp$ is the identity, and the fourth implies that $i^\sharp \circ j^\sharp$ is also the identity. Thus $i^\sharp$ and $j^\sharp$ are isomorphisms, and $H^{k+1}_c(X \times \mathbb{R}) = H^k_c(X)$. In particular, $H^{k+1}_c(\mathbb{R}^{n+1}) = H^k_c(\mathbb{R}^n)$, providing the inductive step of the proof of our theorem.

If $\alpha = \sum \alpha_I(x) dx^I$ is a compactly supported k-form on X, let

$$i(\alpha) = \sum_I \alpha_I(x) f'(s) dx^I \wedge ds = (-1)^k \sum_I f'(s) \alpha_I(x) ds \wedge dx^I,$$

where $f'(s)ds$ is a bump form on $\mathbb{R}$ with integral 1. Let ϕ be the pullback of this bump form to $X \times \mathbb{R}$. Another way of writing the formula for i is then

$$i(\alpha) = \pi_1^*(\alpha) \wedge \phi,$$

where π_1 is the natural projection from $X \times \mathbb{R}$ to X. We check that

$$i(d\alpha) = \pi_1^*(d\alpha) \wedge \phi = d(\pi_1^*(\alpha) \wedge \phi),$$

since $d\phi = 0$.

Next we define j. Every compactly supported k-form α on $X \times \mathbb{R}$ can be written as a sum of two pieces:

$$\alpha = \sum_I \alpha_I(x, s) dx^I + \sum_J \gamma_J(x, s) dx^J \wedge ds,$$

where each I is a k-index and each J is a $(k-1)$-index. We define

$$j(\alpha) = \sum_J \left(\int_{-\infty}^{\infty} \gamma(x, s) ds \right) dx^J.$$

[1]In this construction we work with $X \times \mathbb{R}$ rather than $\mathbb{R} \times X$ to simplify the signs in some of our computations.

Note that

$$\begin{aligned} j(d\alpha) &= j\left(\sum_{I,j} \partial_j \alpha_I(x,s) dx^j \wedge dx^I + \sum_I \partial_s \alpha_I(x,s) ds \wedge dx^I + \sum_{j,J} \partial_j \gamma_J(x,s) dx^j \wedge dx^J \wedge ds\right) \\ &= (-1)^k \sum_I \left(\int_{-infty}^\infty \partial_s \alpha_I(x,s) ds\right) dx^I + \sum_{j,J} \left(\int_{-\infty}^\infty \partial_j \gamma_J(x,s)\right) dx^j \wedge dx^J \\ &= 0 + \sum_{j,J} \partial_j \left(\int_{-\infty}^\infty \gamma_J(x,s)\right) dx^j \wedge dx^J \\ (2) \quad &= d(j(\alpha)), \end{aligned}$$

where we have used the fact that $\alpha_I(x,s)$ is compactly supported, so $\displaystyle\int_{-\infty}^\infty \partial_s \alpha_I(x,s) ds = \alpha_I(x,\infty) - \alpha_I(x,-\infty) = 0$. In the computation of $H^2_c(\mathbb{R}^2)$, the form $\tilde\beta$ was precisely $i \circ j(\beta)$.

Now, for $\alpha \in \Omega^k_c(X \times \mathbb{R})$, let

$$P(\alpha)(x,s) = \sum_J \left(\int_{-\infty}^s \gamma_J(x,t) dt - f(s) \int_{-\infty}^\infty \gamma_J(x,t) dt\right) dx^J.$$

This gives a compactly supported form, since for s large and positive the two terms cancel, while for s large and negative both terms are zero.

Exercise 1: For an arbitrary form $\alpha \in \Omega^k_c(X \times \mathbb{R})$, compute $d(P(\alpha))$ and $P(d\alpha)$, and show that $\alpha - i(j(\alpha)) = \pm dP(\alpha) \pm P(d\alpha)$.

□

2. Computing the top cohomology of compact manifolds

Having established the basic properties of compactly supported forms on $\mathbb{R}^n$, and hence compactly supported forms on a coordinate patch, we consider $H^n(X)$.

Theorem 2.1. *Let X be a compact, connected n-manifold. $H^n(X) = \mathbb{R}$ if X is orientable and is 0 if X is not orientable.*

Proof. We prove the theorem in three steps:

(1) Showing that, if X is oriented, then $H^n(X)$ is at least 1-dimensional.
(2) Showing that, regardless of orientation, $H^n(X)$ is at most 1-dimensional.
(3) Showing that, if X is not oriented, that $H^n(X)$ is trivial.

For the first step, suppose $\alpha \in \Omega^n(X)$. If $\alpha = d\beta$ is exact, then by Stokes' Theorem, $\int_X \alpha = \int_X d\beta = \int_{\partial X} \beta = 0$, since ∂X is the empty set. Since every exact n-form integrates to zero, a closed form that doesn't integrate to zero must represent an non-trivial class in H^n. However, such forms are easy to generate. Pick a point p and a coordinate patch U, and take a bump form of total integral 1 supported on U.

For the second and third steps, we will show that an arbitrary n-form α is *cohomologous* to a finite sum of bump forms, where two closed forms are said to be cohomologous if they differ by an exact form. (That is, if they represent the same class in cohomology.) We then show that any bump form is cohomologous to a multiple of a specific bump form, which then generates $H^n(X)$. If X is not orientable, we will then show that this generator is cohomologous to minus itself, and hence is exact.

Pick a partition of unity $\{\rho_i\}$ subordinate to a collection of coordinate patches. Any n-form α can then be written as $\sum_i \rho_i \alpha_i$. Since X is compact, this is a finite sum. Now suppose that $\alpha_i = \rho_i \alpha$ is compactly supported in the image of a parametrization $\psi_i : U_i \to X$. Let ϕ_i be a bump form on U_i of total integral 1 localized around a point a_i, and let $c_i = \int_{U_i} \psi_i^* \alpha_i$, where we are using the canonical orientation of $\mathbb{R}^n$. Then $\psi_i^* \alpha_i - c_i \phi_i$ is d of a compactly supported $(n-1)$-form on U_i. This implies that $\alpha_i - c_i (\psi_i^{-1})^* \phi_i$ is d of an $(n-1)$-form that is compactly supported on $\psi_i(U_i)$, and can thus be extended (by zero) to be a form on all of X. Thus α_i is cohomologous to $c_i (\psi_i^{-1})^* \phi_i$. That is, to a bump form localized near $p_i = \psi_i(a_i)$.

The choice of a_i was arbitrary, and the precise formula for the bump form was arbitrary, so bump forms with the same integral supported near different points of the same coordinate patch are always cohomologous. However, this means that if ϕ and ϕ' are bump n-forms supported near *any* two points p and p' of X, then ϕ is cohomologous to a multiple of ϕ'. We just find a sequence of coordinate patches $V_i = \psi_i(U_i)$ and points $p_0 = p \in V_1$, $p_1 \in V_1 \cap V_2$, $p_2 \in V_2 \cap V_3$, etc. A bump form near p_0 is cohomologous to a multiple of a bump form near p_1 since both points are in V_1. But that is cohomologous to a multiple of a bump form near p_2 since both p_1 and p_2 are in V_2, etc. Thus α is cohomologous to a sum of bump forms, which is in turn cohomologous to a single bump form centered at an arbitrarily chosen location. This shows that $H^n(X)$ is at most 1-dimensional.

Now suppose that X is not orientable. Then we can find a sequence of coordinate neighborhoods, $V_1, \ldots, V_N$ with $V_N = V_1$, such that there are an odd number of orientation reversing transition functions. Starting with a bump form at p, we apply the argument of the previous paragraph, keeping track of integrals and signs, and ending up with a bump form at $p' = p$ that is *minus* the original bump form. Thus twice the bump form is cohomologous to zero, and the bump form itself is cohomologous to zero. Since this form generated $H^n(X)$, $H^n(X) = 0$.

□

The upshot of this theorem is not only a calculation of $H^n(X) = \mathbb{R}$ when X is connected, compact and oriented, but the derivation of a generator of $H^n(X)$, namely *any* form whose total integral is 1. This can be a bump form, it can be a multiple of the (n-dimensional) volume form, and there are infinitely many other possibilities. The important thing is that integration gives an isomorphism

$$\int_X : H^n(X) \to \mathbb{R}.$$

3. Poincare duality

Suppose that X is an oriented n-manifold (not necessarily compact and not necessarily connected), that α is a closed, compactly supported k-form, and that β is a closed $n-k$ form. Then $\alpha \wedge \beta$ is compactly supported, insofar as α is compactly supported. Integration gives a map

$$\begin{aligned} \int_X : H^k_c(X) \times H^{n-k}(X) &\to \mathbb{R} \\ [\alpha] \times [\beta] &\mapsto \int_X \alpha \wedge \beta. \end{aligned} \tag{3}$$

Exercise 2: Suppose that α and α' represent the same class in $H^k_c(X)$ and that β and β' represent the same class in $H^{n-k}(X)$. Show that $\int_X \alpha' \wedge \beta' = \int_X \alpha \wedge \beta$.

This implies that integration gives a map from $H^{n-k}(X)$ to the dual space of $H^k_c(X)$.

Theorem 3.1 (Poincare duality)**.** *If X is an orientable manifold that admits a finite good cover, then integration gives an isomorphism between $H^{n-k}(X))$ and $(H^k_c(X))^*$.*

Proof. A complete proof is beyond the scope of these notes. A complete presentation can be found in Bott and Tu. Here is a sketch.

- The theorem is true for a single coordinate patch, since $H^{n-k}(\mathbb{R}^n)$ and $H^k_c(\mathbb{R}^n)^*$ are both $\mathbb{R}$ when $k = n$ and 0 otherwise, with integration relating the two as above.
- We construct a Mayer-Vietoris sequence for compactly supported cohomology. With regard to incusions, this runs the opposite direction as the usual Mayer-Vietoris sequence:

  $$\cdots \to H^k_c(U \cap V) \to H^k_c(U) \oplus H^k_c(V) \to H^k_c(U \cup V) \to H^{k+1}(U \cap V) \to \cdots$$

- Looking at dual spaces, we obtain an exact sequence

  $$\cdots \to H^k_c(U \cup V)^* \to H^k_c(U)^* \oplus H^k_c(V)^* \to H^k_c(U \cap V)^* \to H^{k-1}_c(U \cup V)^* \to \cdots$$

  With respect to inclusions, this goes in the same direction as the usual Mayer-Vietoris sequence (going from the cohomology of $U \cup V$ to that of U and V to that of $U \cap V$ to that of $U \cup V$, etc.), only with the index k decreasing at the $U \cap V \to U \cup V$ stage instead of increasing. However, this is precisely what we need for Poincare duality, since decreasing the dimension k is the same thing as increasing the codimension $n-k$.

- The *five lemma* in homological algebra says that if you have a commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

  with the rows exact, and if α, β, δ and ϵ are isomorphisms, then so is γ. Comparing the Mayer-Vietoris sequences for H^{n-k} and for $(H_c^k)^*$, the five lemma says that, if for all k integration gives isomorphisms between $H^{n-k}(U)$ and $H_c^k(U)^*$, between $H^{n-k}(V)$ and $H_c^k(V)^*$, and between $H^{n-k}(U\cap V)$ and $H_c^k(U\cap V)^*$, then integration also induces isomorphisms between $H^{n-k}(U\cup V)$ and $H_c^k(U\cup V)^*$.
- We then use the Mayer-Vietoris argument, proceeding by induction on the cardinality of a good cover.

□

Not-so-hard Exercise 3: Prove the Five Lemma.

Somewhat harder Exercise 4: Set up the Mayer-Vietoris sequence for compactly supported cohomology, using the fact that a compactly supported form on $U\cap V$ can be extended by zero to give a compactly supported form on U or on V, thus defining a map $i:\Omega_c^k(U\cap V)\to\Omega_c^k(U)\oplus\Omega_c^k(V)$, and that we can similarly define a map $j:\Omega_c^k(U)\oplus\Omega_c^k(V)\to\Omega^k(U\cup V)$.

Much harder Exercise 5: Fill in the details of the proof of Poincare duality. The main effort is in setting up the two Mayer-Vietoris sequences and showing that connecting them by integration gives a commutative diagram. You may have to tweak the signs of some of the maps to make this work.

Of course, if X is compact, then there is no difference between H^k and H_c^k. In that case, we have

Corollary 3.2 (Poincare duality for compact manifolds)**.** *If X is a compact oriented manifold, then $H^k(X)$ and $H^{n-k}(X)$ are dual spaces, with a pairing given by integration.*

Exercise 6: Suppose that X is compact, oriented, connected and simply connected. Show that $H^1(X)=0$, and hence that $H^{n-1}(X)=0$. [Side note: The assumption of orientation is superfluous, since all simply connected manifolds are orientable.]

This exercise shows that a compact, connected, simply connected 3-manifold has the same cohomology groups as S^3. The recently-proved Poincare conjecture asserts that all such manifolds are actually homeomorphic to S^3. There are plenty of other (not simply connected!) compact 3-manifolds whose cohomology groups are also the same as those S^3, namely $H^0 = H^3 = \mathbb{R}$ and $H^1 = H^2 = 0$. These are called *rational homology spheres*, and come up quite a bit in gauge theory.

4. Degrees of mappings

Suppose that X and Y are both compact, oriented n-manifolds. Then $H^n(X)$ and $H^n(Y)$ are both naturally identified with the real numbers. If $f : X \to Y$ is a smooth map, then the pullback $f_n^\sharp : H^n(Y) \to H^n(X)$ is just multiplication by a real number. We call this number the *degree of* f. Since homotopic maps induce the same map on cohomology, we immediately deduce that homotopic maps have the same degree.

However, we already have a definition of degree from intersection theory! Not surprisingly, the two definitions agree.

Theorem 4.1. *Let p be a regular value of f, an let D be the number of preimages of p, counted with sign. If $\alpha \in \Omega^n(Y)$, then $\int_X f^*\alpha = D \int_Y \alpha$.*

Proof. First suppose that α is a bump form localized in such a small neighborhood V of p that the stack-of-records theorem applies. That is, f^{-1} is a discrete collection of sets U_i such that f restricted to U_i is a diffeomorphism to V. But then $\int_{U_i} f^*\alpha = \pm \int_V \alpha = \pm \int_Y \alpha$, where the $\pm$ is the sign of $\det f$ at the preimage of p. But then $\int_X f^*\alpha = \sum_i \int_{U_i} f^*\alpha = \sum_i \operatorname{sign}(\det(df)) \int_V \alpha = D \int_Y \alpha$.

Now suppose that α is an arbitrary n-form. Then α is cohomologous to a bump form α' localized around p, and $f^*\alpha$ is cohomologous to $f^*(\alpha')$, so

$$\int_X f^*\alpha = \int_X f^*(\alpha') = D \int_Y \alpha' = D \int_Y \alpha.$$

□

Here is an example of how this shows up in differential geometry. Let X be a compact, oriented 2-manifold immersed (or better yet, embedded) in $\mathbb{R}^3$. At each point $x \in X$, the normal vector $\vec{n}(x)$ (with direction chosen so that $\vec{n}$ followed by an oriented basis for T_xX gives an oriented basis for $T_x(\mathbb{R}^3)$) is a point on the unit sphere. That is, $\vec{n}$ is a map $X \to S^2$. $d\vec{n}$ is then a map from $T_x(X)$ to $T_{\vec{v}(x)}S^n$. But $T_x(X)$ and $T_{\vec{v}(x)}S^n$ are the same space, being the orthogonal complement of $\vec{v}(x)$ in $\mathbb{R}^{n+1}$. Composing $d\vec{v}$ with this identification of $T_x(X)$ and $T_{\vec{v}(x)}S^n$, we get an operator $S : T_x(X) \to T_x(X)$ called the *shape operator* or *Weingarten map*. [Note: Some authors use $-d\vec{v}$ instead of $d\vec{v}$. This amounts to just flipping the sign of

$\vec{v}$.] The eigenvalues of S are called the *principal curvatures* of X at x, and the determinant is called the *Gauss curvature*, and is denoted $K(x)$.

Let ω_2 be the area 2-form on $\S^2$ (explicitly: $\omega_2 = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$). The pullback $\vec{n}^*\omega_2$ is then $K(x)$ times the area form on X.

Exercise 7: Show that the last sentence is true *regardless of the orientation of* X.

The following three exercises are designed to give you some intuition on what K means.

Exercise 8: Let X be the hyperbolic paraboloid $z = x^2 - y^2$. Show that K at the origin is negative, regardless of which orientation you pick for X. In other words, show that $\vec{n}$ is orientation reversing near 0.

Exercise 9: Let X be the elliptic paraboloid $z = x^2 + y^2$. Show that K at the origin is positive.

Exercise 10: Now let X be a general paraboloid $z = ax^2 + bxy + cy^2$, where a and b and c are real numbers. Compute K at the origin. [Or for a simpler exercise, do this for $b = 0$ and arbitrary a and c.]

Theorem 4.2 (Gauss-Bonnet). $\displaystyle\int_X K(x)dA = 2\pi\chi(X)$.

Proof. Since the area of S^2 is 4π, we just have to show that the degree of $\vec{v}$ is half the Euler characteristic of X. First we vary the immersion to put our Riemann surface of genus g in the position used to illustrate the "hot fudge" map. This gives a homotopy of the original map $\vec{v}$, but preserves the degree. Then $(0, 0, 1)$ is a regular value of $\vec{n}$, and has $g+1$ preimages, of which one (at the top) is orientation preserving and the rest are orientation reversing. Thus the degree is $1 - g = \chi(X)/2$. □

This construction, and this theorem, extends to oriented hypersurfaces in higher dimensions. If X is a compact oriented n-manifold in $\mathbb{R}^{n+1}$, we can define the oriented normal vector $\vec{v}(x) \in S^n$, so we still have a map $\vec{v} : X \to S^n$ and a shape operator $S : T_x(X) \to T_x(X)$. The shape operator is always self-adjoint, and so is diagonalizable with real eigenvalues, called the principal curvatures of X, and orthogonal eigenvectors, called the principal *directions*. Let ω_n be the volume form on S^n, and we write $\vec{v}^*\omega_n = K(x)dV$, where dV is the volume form on X. As before, $K(x)$ is called the Gauss curvature of X, and is the determinant of S.

Theorem 4.3 (Gauss-Bonnet in even dimensions). *If X is a compact n-dimensional hypersurface in $\mathbb{R}^{n+1}$, with n even, then* $\displaystyle\int_X K(x)dV = \frac{1}{2}\gamma_n\chi(X)$, *where γ_n is the n-dimensional volume of S^n.*

Proof. As with the 2-dimensional theorem, the key is showing that the degree of $\vec{v}$ is half the Euler characteristic of X. Instead of deforming the immersion of X into a standard form, we'll use the Hopf degree formula.

Pick a point $\vec{a} \in S^n$ such that $\vec{a}$ and $-\vec{a}$ are both regular values of $\vec{v}$. Then the total number of preimages of $\pm\vec{a}$, counted with sign, is twice the degree of $\vec{v}$. We now construct a vector field $\vec{w}(x)$ on X, where $w(x)$ is the projection of $\vec{a}$ onto $T_x(X)$. This is zero precisely where the normal vector is $\pm\vec{a}$.

Exercise 11: Show that the index of $\vec{w}$ at such a point is the sign of $\det(d\vec{v})$.

By the Hopf degree formula, the Euler characteristic of X is then twice the degree of $\vec{v}$. Since $\int_X K(x)dV = \int_X \vec{v}^*\omega_n = \frac{\text{Degree}}{2}\gamma_n$, the theorem follows.

□

If X is odd-dimensional, then $\chi(X) = 0$, but $\int_X K(x)dV$ need not be zero. A simple counter-example is S^n itself.

Another application of degrees and differential forms comes up in knot theory. If X and Y are non-intersecting oriented loops in $\mathbb{R}^3$, then there is a map $f(X \times Y) \to S^2$, $f(x,y) = (y-x)/|y-x|$. The *linking number* of X and Y is the degree of this map. This can be computed in several ways. One is to pick a point in S^2, say (0,0,1), and count preimages. These are the instances where a point in Y lies directly above a point in X. In other words, the linking number counts crossings with sign between the knot diagram for X and the knot diagram for Y. However, it can also be expressed via an integral formula.

Exercise 12: Supposed that γ_1 and γ_2 are non-intersecting loops in $\mathbb{R}^3$, where each γ_i is, strictly speaking, a map from S^1 to $\mathbb{R}^3$. Then f is a map from $S^1 \times S^1$ to S^2. Compute $f^*\omega_2$ in terms of $\gamma_1(s)$, $\gamma_2(t)$ and their derivatives. By integrating this, derive an integral formula for the linking number.