Differential Topology
Homework 1: Due January 29

There will be weekly homework assignments due each Friday at the *beginning* of class. Try the problems on your own first. Then feel free to discuss them and work together with classmates, friends, pets, favorite oracle, etc. However, I expect you to write up your own solutions to the problems.

Problem 1. Prove the Inverse Function Theorem in one dimension: "If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and if $f'(a) \neq 0$, then there exists a neighborhood U of a and V of $f(a)$ such that 1) f maps U in a 1–1 manner onto V, 2) $f^{-1} : V \to U$ is differentiable at a, and 3) $(f^{-1})'(f(a)) = 1/f'(a)$."

Problem 2. Let g be a continuous real-valued function on the unit circle in $\mathbb{R}^2$ such that $g(0,1) = g(1,0) = 0$ and $g(-x_1, -x_2) = -g(x_1, x_2)$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = |x|g(x/|x|)$ if $x \neq 0$ and $f(0) = 0$. (Here $x \in \mathbb{R}^2$ is shorthand for (x_1, x_2).)
(a) Show that f is continuous at $(0,0)$, that the partial derivatives of f at $(0,0)$ are well-defined, and that the directional derivatives of f in every direction are well-defined.
(b) Show that f is not differentiable at (0,0) unless g is identically zero.

Problem 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x|y|/\sqrt{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. (This is a function of the kind considered in Problem 3.) Compute the partial derivatives of f on a neighborhood of the origin, and show that they are well-defined everywhere but are not continuous at the origin.

Problem 4. Let $x \in \mathbb{R}$. A *derivation at* x is a map $D : C^\infty(\mathbb{R}) \to \mathbb{R}$ such that, for any smooth functions f, g and any scalar c,

$$\begin{aligned} D(f+g) =& D(f) + D(g) \\ D(cf) =& cD(f) \\ D(fg) =& f(x)D(g) + g(x)D(f) \end{aligned}$$

Let V_x be the set of derivations at x.

(a) Show that d/dx, followed by evaluation at a point x, is a derivation at x.
(b) Show that V_x is a vector space, with the obvious notions of addition and scalar multiplication.
(c) Show that V_x is one-dimensional, with basis d/dx. (Hint: Use Taylor's theorem).
(d) Let V be the disjoint union of all the sets V_x. Define a reasonable topology for V such that V is connected.

Problem 5. Now generalize to $\mathbb{R}^n$. A derivation at x is, as before, a linear map $D : C^\infty(\mathbb{R}^n) \to R$ that satisfies the above axioms. Show that V_x is an n-dimensional vector space with basis $\{\partial/\partial x_i\}$.

The upshot of problems 4 and 5 is that the tangent space to $\mathbb{R}^n$ can be defined purely algebraically from the ring of smooth functions on $\mathbb{R}^n$. Since this is a local operation, the tangent space of an arbitrary smooth manifold can be similarly constructed from the ring of smooth functions on that manifold. That's not an angle we will be emphasizing in this class, but it's at the heart of non-commutative geometry, and also of algebraic geometry (with "smooth" replaced by "analytic" or "algebraic").

Problem 6: Let A_0 be an $m \times n$ matrix of maximal rank. Show that there is a neighborhood U of A_0 in the space of $m \times n$ matrices (i.e. of $\mathbb{R}^{nm}$) and an algorithm for finding a basis for the kernel (for $m \leq n$) and image (for $m \geq n$) of each $A \in U$ such that these bases vary continuously with the entries of A. In other words, not only is the condition "A is of maximal rank" stable, but the kernels and images don't change much as we vary A.