Differential Topology

Homework 6: Due March 4

Problem 1. Suppose that f is a continuous function on $\mathbb{R}^n$ and g is a smooth function of compact support. Show that the convolution $f * g(x) = \int f(y)g(x-y)\, d^n y$ is smooth.

Problem 2. Suppose that C is a compact set in $\mathbb{R}^n$ and that U is an open set containing C. Show that there exists a smooth function $f : \mathbb{R}^n \to [0,1]$ such that $\phi(x) = 1$ for $x \in C$ and such that the support of f (a.k.a. the smallest *closed* set containing all points where $f(x) > 0$) is contained in U.

Recall that a *partition of unity* on a set S subbordinate to an open cover $\{U_i\}$ of S is a collection of smooth functions $\{f_j\}$ such that:

1. Each f_j takes values between 0 and 1 (inclusive).
2. The support of each f_j is contained in a single U_i.
3. Every point in S has a neighborhood on which all but a finite number of the f_j's are zero.
4. $\sum_j f_j(x) = 1$ for all $x \in S$.

Note that the indices i and j may live in different sets. Every function f_j has support contained in some open set U_i, but it may NOT be true that every U_i has a function supported in it.

The next few problems concern construction of partitions of unity on subsets of $\mathbb{R}^n$. The phrase "subordinate to the open cover $\{U_i\}$" is always assumed. Also, a sequence of subsets S_i is said to be *nested* if the closure of each S_i is contained in the interior of S_{i+1}.

Show that a partition of unity exists

Problem 3. When S is a compact subset of $\mathbb{R}^n$.

Problem 4. When S is the union of a sequence $\{C_i\}$ of nested compact subsets of $\mathbb{R}^n$. [Hint: construct functions whose support lie in $C^0_{i+2} - C_i$. If you need more help, check the argument on page 53.]

Problem 5. When S is an open set in $\mathbb{R}^n$. [Hint: Express S the union of a nested sequence of compact sets.]

Problem 6. When S is an arbitrary subset of $\mathbb{R}^n$.

The upshot is that we can always construct partitions of unity on concrete manifolds, insofar as they are subsets of some $\mathbb{R}^N$. But what about abstract manifolds?

Problem 7. Suppose X is an abstract connected k-manifold, with open cover $\{U_i\}$. Show that X admits a partition-of-unity. [Hint: Use the fact that the open subsets of X have a countable basis consisting of neighborhoods diffeomorphic to balls in $\mathbb{R}^k$. Use this countability to show that X is the union of a countable sequence of nested compact sets. Then apply an argument similar to the argument of problem 4.]

Now that we have constructed partitions of unity for abstract manifolds, let's prove the Transversality Homotopy Theorem (page 70) for abstract manifolds. Assume X and Y are manifolds, with X compact, and that Z is a closed submanifold of Y. Given smooth $f : X \to Y$, we want to show that f is homotopic to some $g : X \to Y$ with $g \pitchfork Z$. Likewise, if X is a compact manifold-with-boundary, then we want to construct a family of maps f_s such that for almost every s, $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$. Since Y is an abstract manifold, we cannot use the ϵ-neighborhood $Y^\epsilon \subset \mathbb{R}^N$. Instead, we use vector fields.

Problem 8. For each compact subset C of Y, show that there exist a finite collection $\{v_i\}$ of vector fields, each of compact support, such that, for all y in some neighborhood U of C, $\{v_i(y)\}$ spans $T_y(Y)$.

A smooth vector field on a manifold induces a *flow*. Given a vector field v, let $\phi_{v,t}(x_0)$ be the solution, at time t, to the differential equation $dx/dt = v(x)$ with initial condition $x(0) = x_0$. Standard theorems about differential equations imply that this equation has a unique solution for short time. If we also impose growth conditions on v, then the solution exists for all time. In particular, if v has compact support, then the flow is defined for all time.

Now suppose that $C = f(X)$. Find a finite collection of N vector fields as in Problem 8. Let S be an open set in $\mathbb{R}^N$, and define

$$F(s, x) = \phi_{v_1,s_1} \circ \phi_{v_2,s_2} \circ \cdots \circ \phi_{v_N,s_N}(f(x)).$$

Problem 9. Show that, if S is chosen to be a small enough ball in $\mathbb{R}^N$ (centered at the origin), then F (and ∂F if X has boundary) are submersions, and in particular are transversal to Z. By the Thom Transversality Theorem, this implies that f_s and ∂f_s are transversal to Z for almost every s.