M382D Midterm Exam Solutions, March 11, 2016

Do **three** out of four problems. (The fourth is on the back.) Be clear about which problems you want graded. If you attempt all four problems, I'll just grade the first three.

1a) Show that CP^1 is a manifold by explicitly constructing a smooth atlas.

1b) The linear map $C^2 \to C^2$ given by the matrix $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ sends 1-dimensional subspaces of C^2 to 1 dimensional subspaces of C^2, and so induces a map $f : CP^1 \to CP^1$. Let $x \in CP^1$ be the line spanned by $(1, 1)$. Compute the matrix of df_x with respect to coordinates that you already defined in part (a).

a) Every complex subspace (a.k.a. complex line) has a basis vector (z_0, z_1). If $z_0 \neq 0$ we rescale this to $(1, z_1/z_0)$ and use coordinates (x, y) where $z_1/z_0 = x + iy$. If $z_1 \neq 0$ we rescale the basis vector to $(z_0/z_1, 1)$ and use coordinates (u, v) where $z_0/z_1 = u + iv$. On the overlap of these charts we have $u + iv = 1/(x + iy)$, so $u = x/(x^2 + y^2)$ and $v = -y/(x^2 + y^2)$, or equivalently $x = u/(u^2 + v^2)$ and $y = -v/(u^2 + v^2)$. This gives a smooth change-of-coordinates function (in either direction).

b) Our linear map sends $\begin{pmatrix} 1 \\ z \end{pmatrix}$ to $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 + iz \\ i + z \end{pmatrix}$, and so sends the line spanned by $(1, z)$ to a line spanned by $(1, \frac{i+z}{1+iz})$. Working in our first coordinate chart (where $z_1/z_0 = x+iy$), and abusing notation by writing "f" in place of $\phi^{-1} \circ f \circ \phi$, where $\phi : \mathbb{R}^2 \to CP^1$ is our coordinate parametrization, we expand $(i + z)/(1 + iz)$ and take real and imaginary parts to get

$$f(x, y) = \left(\frac{2x}{(y-1)^2 + x^2}, \frac{1 - x^2 - y^2}{(y-1)^2 + x^2} \right).$$

Taking the matrix of partial derivatives at $(x, y) = (1, 0)$ then gives $\partial f_1/\partial x = \partial f_2/\partial y = 0$, $\partial f_1/\partial y = 1$ and $\partial f_2/\partial x = -1$, so the matrix of $df_{(1,0)}$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2) Let X be a manifold (without boundary) of arbitrary dimension, and let $f : X \to \mathbb{R}^2$ be an arbitrary smooth map. Show that there exists a line $L \subset \mathbb{R}^2$ (not necessarily through the origin) such that $f \pitchfork L$.

Consider vertical lines L_a of the form $x = a$. Since L_a is a level set of the function $g(x, y) = x$, the statement $f \pitchfork L_a$ is equivalent to "a is a regular value of $g \circ f$". (This observation is how we came up with the definition of transversality in the first place!) But almost every $a \in \mathbb{R}$ is a regular value of $g \circ f$, by Sard's theorem. So f is transversal to almost every vertical line. (And almost every line in any other direction, for similar reasons).

3) Let K be the Klein bottle, obtained by identifying edges of the unit square as in the figure below. Let C be the subset $x = 1/2$, where (x, y) are the usual coordinates on the unit square. **Using intersection theory**, show that the inclusion $i_C : C \to K$ is not homotopic to a constant map. (There are lots of ways to do this using the machinery of algebraic topology, such as knowing that C is one of the standard generators of the fundamental group of K. Those do NOT count. Do this using the machinery we developed THIS semester.)

Let Z be the curve $y = 1/2$. This is a closed submanifold of K. Then $C \pitchfork Z$ and there is one intersection point, namely (1/2,1/2), so $I_2(C, Z) = 1$. But if i_C was null-homotopic, then i_C would be homotopic to a map that missed Z altogether, and we would have $I_2(C, Z) = 0$. Contradiction.

4) Let $X \subset \mathbb{R}^n$ be a compact manifold, $Y \subset \mathbb{R}^m$ a compact manifold, and Z_0 and Z_1 closed sub-manifolds of Y. Suppose that $f_0 \pitchfork Z_0$, that $f_1 \pitchfork Z_1$, and that f_0 and f_1 are homotopic. Show that there is a smooth map $G : [0, 1] \times X \to Y$ with the following properties:

- The restriction of G to $(0, 1) \times X$ is transversal to both Z_0 and Z_1.
- $g_0 = f_0$ (where $g_0(x) := G(0, x)$).
- $g_1 = f_1$.

(Note: we can't require that G itself be transversal to both Z_0 and Z_1 since this would involve properties of f_0 and f_1.)

This is a small variation on the proof of the Transversality Homotopy Theorem, along the same lines as the Extension Theorem but without having to deal with partitions of unity. Since Y is compact, it admits an ϵ neighborhood Y^ϵ. Let $\pi : Y^\epsilon \to Y$ be the projection, and take S to be the ball of radius ϵ in $\mathbb{R}^m$. If F is the homotopy between f_0 and f_1, define $\hat{F} : S \times [0, 1] \times X \to Y$ by

$$\hat{F}(s, t, x) = \pi(F(t, x) + t(1 - t)s).$$

When restricted to $S \times (0,1) \times X$ this is a submersion, hence transversal to both Z_0 and Z_1. By the Thom Transversality Theorem, for almost every $s \in S$ we have $\hat{F}_s \pitchfork Z_0$ (when restricted to $(0,1) \times X$) and for a.e. s, $\hat{F}_s \pitchfork Z_1$ (when restricted to $(0,1) \times X$), so for almost every s, $\hat{F}_s$ is transversal to both Z_0 and Z_1. (The intersection of two sets of full measure has full measure.) Pick such an s and define $G(t,x) = \hat{F}(s,t,x)$. Because of the factor of $t(1-t)$, we have $G(0,x) = F(0,x) = f_0(x)$ and $G(1,x) = F(1,x) = f_1(x)$, so $g_0 = f_0$ and $g_1 = f_1$.

Note, by the way, that it doesn't make sense to speak of being transversal to $Z_0 \cup Z_1$, since $Z_0 \cup Z_1$ needn't be a manifold. We can only speak of being transversal to Z_0 and transversal to Z_1.