M408D Final Exam Solutions, May 14, 2016

1) Methods of integration. Compute the following integrals:

a) $\int x^2 \ln(x)dx$.

Integrate by parts with $u = \ln(x)$ and $dv = x^2 dx$. The result is then $\frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C$.

b) $\int_2^3 \frac{x-2}{x^3 - x^2}dx$.

Use partial fractions to convert $\frac{x-2}{x^3-x^2}$ to $\frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-1}$. The integral then gives $\ln(3) - 2\ln(2) + \frac{1}{3}$.

c) $\int_0^\pi \cos(x)\sin(2x)dx$

There are several ways to do this. You can integrate by parts twice using the "going in circles" trick. You can use an addition of angle formula to replace the integrand with $\frac{1}{2}(\sin(3x) + \sin(x))dx$. Or you can use $\sin(2x) = 2\cos(x)\sin(x)$ to write the integral as $\int_0^\pi 2\sin(x)\cos^2(x)dx = -\frac{2\cos^3(x)}{3}\Big|_0^\pi$. However you do it, the answer is $4/3$.

2. The outline of a fan blade is given by the polar curve $r = 2\sin(3\theta)$, where θ ranges from 0 to $\pi/3$.

a) Compute the area of the blade.

$$
\begin{aligned}
\text{Area} &= \int_0^{\pi/3} \frac{r^2}{2}d\theta \\
&= \int_0^{\pi/3} 2\sin^2(3\theta)d\theta \\
&= \int_0^{\pi/3} 1 - \cos(6\theta)d\theta \\
&= \theta - \frac{\sin(6\theta)}{6}\Big|_0^{\pi/3} = \frac{\pi}{3}.
\end{aligned}
$$

b) Suppose that the fan blade has constant density $\rho(x,y) = 1$. Compute the moment of inertial I_z of the fan blade (for rotation about the z axis).

$$
I_z = \int_0^{\pi/3} \int_0^{2\sin(3\theta)} \rho r^2 r\,dr\,d\theta
$$

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$$= \int_0^{\pi/3} \int_0^{2\sin(3\theta)} r^3 \, dr \, d\theta$$

$$= \int_0^{\pi/3} 4\sin^4(3\theta) \, d\theta$$

$$= \int_0^{\pi/3} (1 - \cos(6\theta))^2 \, d\theta$$

$$= \int_0^{\pi/3} 1 - 2\cos(6\theta) + \frac{1}{2}(1 + \cos(12\theta)) \, d\theta$$

$$= \frac{3\theta}{2} - \frac{\sin(6\theta)}{3} + \frac{\sin(12\theta)}{24} \Big|_0^{\pi/3} = \pi/2,$$

where we have repeatedly used the double angle formulas $2\sin^2(A) = 1 - \cos(2A)$, $2\cos^2(A) = 1 + \cos(2A)$.

c) Write down an integral that gives the perimeter of the blade. **You do not have to evaluate this integral** (which can't be done in closed form), but you should simplify the integrand as much as possible, and you should be explicit about the limits of integration.

$$\text{Length} = \int_0^{\pi/3} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$$

$$= \int_0^{\pi/3} 2\sqrt{\sin^2(3\theta) + 9\cos^2(3\theta)} \, d\theta$$

$$= \int_0^{\pi/3} 2\sqrt{1 + 8\cos^2(3\theta)} \, d\theta.$$

3. Differential equations. Solve the following differential equations, with the given initial conditions.

a) $\dfrac{dy}{dx} = \dfrac{\cos(x)}{2y}$, $y(0) = 2$.

This is separable. $2y \, dy = \cos(x) \, dx$, so $y^2 = \sin(x) + C$. Plugging in $y(0) = 2$ gives $C = 4$, so $y = \sqrt{\sin(x) + 4}$. (Note that we want the positive square root, since $y(0)$ is positive.)

b) $\dfrac{dy}{dx} + \dfrac{2xy}{x^2 + 1} = 3$; $y(1) = 3$.

This is linear. The integrating factor is $\exp(\int \frac{2x \, dx}{x^2 + 1}) = \exp(\ln(x^2 + 1)) = x^2 + 1$. We get

$$(x^2 + 1)y' + 2xy = 3(x^2 + 1)$$

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$$\begin{aligned}
[(x^2+1)y]' &= 3x^2+3 \\
(x^2+1)y &= x^3+3x+C \\
2(3) &= 1^3+3(1)+C, \text{ so } C=2 \\
y &= \frac{x^3+3x+2}{x^2+1}
\end{aligned}$$

4. Convergence. Which of the following series converge absolutely? Which converge conditionally? Which diverge? **WHY?** (You must explain which test you are using and what it says to receive full credit.)

a) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{\sqrt{k^5+17}}$

This converges by limit comparison to $\sum k^{-3/2}$. Since the terms are already positive, it converges absolutely.

b) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(n)^n}$

This converges absolutely by the root test. $\lim |a_n|^{1/n} = \lim \frac{1}{\ln(n)} = 0$.

c) $\displaystyle\sum_{m=0}^{\infty} m!e^{-5m}$.

This diverges by the ratio test. $a_{m+1}/a_m = (m+1)e^{-5} \to \infty$.

d) $\displaystyle\sum_{r=1}^{\infty} \left(1-\frac{1}{r}\right)^{2r}$

This diverges by the divergence test. The individual terms approach e^{-2}, not 0.

5. Power series. Consider the function

$$f(x) = 2 + (1+x)\ln(1+x) + (1-x)\ln(1-x).$$

a) Express $f''(x)$ as an infinite power series (in powers of x). [Hint: First compute $f''(x)$, and use what you know about geometric series.] What is the interval of convergence of this series?

$$\begin{aligned}
f(x) &= 2 + (1+x)\ln(1+x) + (1-x)\ln(1-x) \\
f'(x) &= \ln(1+x) + 1 - \ln(1-x) - 1 = \ln(1+x) - \ln(1-x) \\
f''(x) &= \frac{1}{1+x} + \frac{1}{1-x}
\end{aligned}$$

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$$= \frac{2}{1 - x^2}$$

$$= \sum_{n=0}^{\infty} 2x^{2n}$$

This is a geometric series that converges for $-1 < x < 1$. It diverges at $x = \pm 1$. If you didn't notice that $\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$, you can just take the geometric series $\frac{1}{1+x} = \sum_{k=1}^{\infty}(-1)^k x^k$ and $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ and add them up. The odd terms cancel, and the even terms (with $k = 2n$) give the same as above.

b) Now express $f'(x)$ as a power series. What is the interval of convergence?

Integrate the series for $f''(x)$ term by term:

$$f'(x) = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n + 1}.$$

Since $f'(0) = 0$, C must be zero. This series converges for $-1 < x < 1$. This is the same interval as for f''. When $x = \pm 1$ the series diverges by limit comparison to $\sum 1/n$. It is *not* an alternating series when $x = -1$, since we only get odd powers of x.

c) Finally, express $f(x)$ as a power series. What is the interval of convergence?

Integrating again gives

$$f(x) = C' + \sum_{n=0}^{\infty} \frac{2x^{2n+2}}{(2n + 1)(2n + 2)}.$$

Since $f(0) = 2$, we have $C' = 2$. This series converges for $-1 \le x \le 1$. For $x = \pm 1$ it converges by limit comparison to $\sum 1/n^2$. It has the same *radius* of convergence as the series for f' and f'', but a slightly different *interval* of convergence.

6. Taylor polynomials.

a) Find the 3rd order Taylor polynomial for $f(x) = x^2 e^{2-x}$ around $x = 2$.

The first step is to compute some derivatives and evaluate them at $x = 2$:

$$\begin{aligned}
f(x) &= x^2 e^{x-2}; & f(2) &= 4 \\
f'(x) &= (2x - x^2)e^{x-2}; & f'(2) &= 0 \\
f''(x) &= (x^2 - 4x + 2)e^{x-2}; & f''(2) &= -2
\end{aligned}$$

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$$f'''(x) = (-x^2 + 6x - 6)e^{x-2}; \qquad f'''(2) = 2$$

So our 3rd order Taylor polynomial is

$$T_3(x) = 4 + 0(x-2) - (x-2)^2 + \frac{1}{3}(x-2)^3.$$

b) Use this to approximate $f(2.1)$ to 4 decimal places.

Plugging in $x = 2.1$ gives

$$f(2.1) \approx T_3(2.1) = 4 - 0.01 + 0.000333 = 3.990333\ldots$$

7. Partial derivatives and parametrized curves (2 pages!). Consider the function
$$f(x,y) = 2ye^{x-1} - xe^{y-1}.$$

[Note: there was a typo in this problem. The function was supposed to be $2ye^{x-1} - xe^{y-2}$, which would have made the answers come out very cleanly. Here is the solution to the problem as it actually appeared.]

a) Compute the partial derivatives f_x and f_y (as functions of x and y) and evaluate these functions at $(x,y) = (1,2)$.

$$\begin{aligned}
f(x,y) &= 2ye^{x-1} - xe^{y-1}; & f(1,2) &= 4 - e \\
f_x(x,y) &= 2ye^{x-1} - e^{y-1}; & f_x(1,2) &= 4 - e \\
f_y(x,y) &= 2e^{x-1} - xe^{y-1}; & f_y(1,2) &= 2 - e
\end{aligned}$$

b) Use these to approximate $f(1.01, 2.02)$.

$$\begin{aligned}
f(1.01, 2.02) &\approx f(1,2) + 0.01 f_x(1,2) + 0.02 f_y(1,2) \\
&= (4 - e) + 0.01(4 - e) + 0.02(2 - e) \\
&= 4.08 - 1.03e
\end{aligned}$$

c) Suppose that we are moving along a parametrized curve $x = t^2$, $y = 1+t^3$. Compute the slope of this curve at $t = 1$. (That is, when $x = 1$ and $y = 2$)

$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{3t^2}{2t}\right|_{t=1} = \frac{3}{2}.$$

d) Compute $\dfrac{d}{dt}f\big(x(t), y(t)\big)$ at $t = 1$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (4 - e)(2) + (2 - e)(3) = 14 - 5e.$$

8. Multiple integrals. Consider the iterated Type I integral

$$I = \int_0^1 \int_{x^{1/2}}^{x^{1/3}} \frac{1}{y}dy\,dx.$$

a) Sketch the region of integration.

This is the region between the curves $y = x^{1/2}$ (in other words $x = y^2$) and $y = x^{1/3}$ (in other words $x = y^3$), which intersect at $(0,0)$ and $(1,1)$. Note that $y = x^{1/3}$ lies to the left (or above) $y = x^{1/2}$.

b) Rewrite I as a Type II integral.

$$\int_0^1 \int_{y^3}^{y^2} \frac{1}{y}dx\,dy$$

c) Evaluate this Type II integral.

The integral over x gives $\frac{y^2 - y^3}{y} = y - y^2$, so we are left with

$$\int_0^1 (y - y^2)dy = \frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1 = \frac{1}{6}.$$