M346 First Midterm Exam, September 22, 2005

1. Let $V = \mathbb{R}_2[t]$ with (standard) basis $\mathcal{B} = \{1, t, t^2\}$ and let $W = M_{2,2}$ be the space of 2 by 2 real matrices with (standard) basis $\mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Consider the linear transformation $L(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(1) - \mathbf{p}(0) & \mathbf{p}(2) - \mathbf{p}(0) \\ \mathbf{p}(-1) - \mathbf{p}(0) & \mathbf{p}(-2) - \mathbf{p}(0) \end{pmatrix}$ from V to W.

a) Find the matrix of L relative to the bases $\mathcal{B}$ and $\mathcal{D}$.

b) What is the dimension of $Ker(L)$? Find a basis for $Ker(L)$.

c) What is the dimension of $Range(L)$? Find a basis for $Range(L)$.

2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{pmatrix}$.

a) Let $V = \{\mathbf{x} \in \mathbb{R}^4 | A\mathbf{x} = 0\}$. What is the dimension of V? Find a basis for V.

b) In $\mathbb{R}^3$, consider the vectors $(1, 2, 5)^T$, $(2, 4, 10)^T$, $(3, 5, 13)^T$, and $(4, 7, 18)^T$. Are these vectors linearly independent? Do they span $\mathbb{R}^3$?

c) Find a basis for the span of the four vectors of part (b).

3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{pmatrix}$. On $\mathbb{R}^2$, consider the standard basis $\mathcal{E}$ and the alternate basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\}$. Finally, let $\mathbf{v} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$.

a) Find $P_{\mathcal{E}\mathcal{B}}$, $P_{\mathcal{B}\mathcal{E}}$, $[\mathbf{v}]_{\mathcal{E}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

b) Find the matrix $[L]_{\mathcal{E}}$ and the matrix $[L]_{\mathcal{B}}$.

4. The two parts of this problem are NOT connected.

a) In $\mathbb{R}_2[t]$, consider the vectors $\mathbf{b}_1 = 1 + t + 2t^2$, $\mathbf{b}_2 = 2 + 3t + 5t^2$ and $\mathbf{b}_3 = 3 + 7t + 9t^2$. Do these vectors form a basis for $\mathbb{R}_2[t]$? If so, find $[\mathbf{v}]_{\mathcal{B}}$, where $\mathbf{v} = 1 - 2t$. If not, find constants a_1, a_2, a_3, not all zero, such that $a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + a_3\mathbf{b}_3 = 0$.

b) In $\mathbb{R}_3[t]$, let V be the set of polynomials $\mathbf{p}$ for which $\mathbf{p}(0) = \mathbf{p}(1) = 0$. Find a basis for V.

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) The plane $x_1 + 3x_2 - 4x_3 = 0$ is a subspace of $\mathbb{R}^3$.

b) If A is a 3×5 matrix, then the dimension of the null space of A is at least 2.

c) Let $L : \mathbb{R}_5[t] \to \mathbb{R}^3$ be a linear transformation. If L is onto, then the kernel of L is 2-dimensional.

d) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. If n vectors $\mathbf{d}_1, \ldots, \mathbf{d}_n$ span V, then the vectors $[\mathbf{d}_1]_{\mathcal{B}}, \ldots, [\mathbf{d}_n]_{\mathcal{B}}$ are linearly independent.

e) Every linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^4$ is multiplication by a 5×4 matrix.