M346 First Midterm Exam Solutions, February 11, 2009

1a) In $\mathbb{R}^3$, let $\mathcal{E}$ be the standard basis and let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be an alternate basis. Let $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 14 \end{pmatrix}$. Find $P_{\mathcal{E}\mathcal{B}}$, $P_{\mathcal{B}\mathcal{E}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

$P_{\mathcal{E}\mathcal{B}} = (\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$, $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$, and $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}\mathbf{v} = \begin{pmatrix} 3 \\ -8 \\ 37 \end{pmatrix}$.

1b) In $\mathbb{R}_2[t]$, let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis and let $\mathcal{B} = \{1 + 2t + 3t^2, t + 4t^2, t^2\}$ be an alternate basis, and let $\mathbf{v} = 3 - 2t + 14t^2$. Find $P_{\mathcal{E}\mathcal{B}}$, $P_{\mathcal{B}\mathcal{E}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

This is essentially the same problem as 1a, and has the exact same answers.

2. Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + 3x_3 \\ x_1 + 3x_2 + 5x_3 \\ x_1 + 4x_2 + 7x_3 \end{pmatrix}$.

a) Find the matrix of L (relative to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^4$.

$$[L]_{\mathcal{E}} = (L(\mathbf{e}_1)L(\mathbf{e}_2)L(\mathbf{e}_3)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix}.$$

b) Let $V = \{\mathbf{x} \in \mathbb{R}^3 : L(\mathbf{x}) = 0\}$. What is the dimension of V? Find a basis for V.

Row-reducing $[L]_{\mathcal{E}}$ *gives* $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Since there is one free variable,* V *is one dimensional, and a basis is* $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

3. On $\mathbb{R}^2$, consider the basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$. Let $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 19 & -30 \\ 10 & -16 \end{pmatrix}$.

a) Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$, where $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is the standard basis.

Much as in problem 1, $P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ *and* $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. *(You can get that by row reduction or by the formula for the inverse of a* 2×2 *matrix.)*

b) Find the matrix of L relative to the $\mathcal{B}$ basis.

$[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

c) If we were given the problem of solving the evolution equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$, we would switch to coordinates $\mathbf{y} = [\mathbf{x}]_{\mathcal{B}}$. Rewrite the equations in terms of the variables y_1 and y_2. You do *not* need to solve these equations for $\mathbf{y}(n)$ in terms of $\mathbf{y}(0)$. Just get $\mathbf{y}(n+1)$ in terms of $\mathbf{y}(n)$.

$\mathbf{y}(n+1) = [L]_{\mathcal{B}}\mathbf{y}(n)$, *so* $y_1(n+1) = 4y_1(n)$ *and* $y_2(n+1) = -y_2(n)$.

4. True of False? Each question is worth 5 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If four vectors in $\mathbb{R}_3[t]$ are linearly independent, then they form a basis for $\mathbb{R}_3[t]$.

True, since $\mathbb{R}_3[t]$ *is 4-dimensional.*

b) If A is a 3×5 matrix whose rank is two, then the set of solutions to $A\mathbf{x} = 0$ is a 2-dimensional subspace of $\mathbb{R}^5$.

False. The dimension of the null space is 3.

c) If $\mathcal{B}$ and $\mathcal{D}$ are basis for a vector space V, then the change-of-basis matrices $P_{\mathcal{B}\mathcal{D}}$ and $P_{\mathcal{D}\mathcal{B}}$ are inverses of one another.

True.

d) If $P_{\mathcal{B}\mathcal{D}}$ is a change-of-basis matrix, then for any vector $\mathbf{v}$, $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{D}\mathcal{B}}[\mathbf{v}]_{\mathcal{D}}$.

False. $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{D}}[\mathbf{v}]_{\mathcal{D}}$.

e) The columns of an $m \times n$ matrix A span $\mathbb{R}^m$ if and only the reduced row-echelon form of A has a pivot in each column.

False. You need a pivot in each row, not a pivot in each column.