M346 Final Exam, August 15, 2011

1) The 4×4 matrix $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 7 & 1 & -24 \\ 1 & 1 & 4 & 9 \\ 4 & 8 & 1 & -29 \end{pmatrix}$ row-reduces to $B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Let $\mathbf{a}_i$ denote the $i$the column of A.

a) Write one of the columns of A explicitly as a linear combination of the others. (I'm looking for something like "$\mathbf{a}_3 = \mathbf{a}_1 + 7\mathbf{a}_2 - 12\mathbf{a}_4$", although that's not actually the right answer.)

b)Find a basis for the null space of A.

c) Find a basis for the column space of A.

2. Let V be a 2-dimensional vector space, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be another basis, with $\mathbf{d}_1 = \mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{d}_2 = 3\mathbf{b}_1 + 4\mathbf{b}_2$.

a) Compute the change-of-basis matrices $P_{\mathcal{B}\mathcal{D}}$ and $P_{\mathcal{D}\mathcal{B}}$.

b) Let $\mathbf{v} = 4\mathbf{b}_1 - 2\mathbf{b}_2$. Compute $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{D}}$.

c) Let $L : V \to V$ be an operator. If $[L]_{\mathcal{D}} = \begin{pmatrix} 0 & -12 \\ 2 & 10 \end{pmatrix}$, what is $[L]_{\mathcal{B}}$? (Note the subscripts! We are converting from $\mathcal{D}$ to $\mathcal{B}$ coordinates, not the other way around.)

3. Diagonalization. Consider the matrix $A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Compute e^{At}.

c) If $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ and $\mathbf{x}(0) = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$, find $\mathbf{x}(t)$ for all t.

4. Down the rabbit hole. The folks at Texas A& M have developed a type of rabbit that breeds even faster than regular rabbits. (Why they would want to do this is beyond me.) Every month, each baby female Aggie Rabbit grows to adulthood. Every month, each adult female Aggie Rabbit gives birth to 6 female babies. (We are only keeping track of female rabbits, assuming that there are enough males around to Do Their Thing.)

a) Set up the equations that govern the evolution of the rabbit population. (I am *not* asking for the solution in this part. Just the evolution equations themselves.)

b) If we start with 5 (female) babies in month 0, how many adults and how many babies will we have in month n?

c) After many months, what will be the approximate ratio of adult to baby rabbits?

5. Consider the nonlinear difference equations

$$\begin{aligned} x_1(n+1) &= \frac{x_1(n)^3}{5x_2(n)^4} + \frac{4}{5} \\ x_2(n+1) &= \frac{1}{5}\left(e^{3(x_1(n)x_2(n)-1)} + x_1(n) + 3\right) \end{aligned}$$

a) Find a system of linear difference equations that approximates this non-linear system near the fixed point $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Your answer should be an explicit set of equations, of the general form "quantity related to $\mathbf{x}(n+1) \approx$ linear function of same quantity computed from $\mathbf{x}(n)$."

b) Determine whether the fixed point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is stable, unstable, or borderline.

6. Find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 4 & 7 \\ -1 & 0 & -13 \\ 1 & 7 & 10 \\ -1 & -1 & 26 \end{pmatrix}$, where we are defining "orthogonal" using the usual inner product on $\mathbb{R}^4$.

7. Find the equation of the best line through the four points (0,0), (1,1), (2,4), and (3,9). (Never mind that this is obviously a parabola. Find the best*line*.)

8. Consider the wave equation $\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\partial^2 f(x,t)}{\partial x^2}$ with velocity $v = 1$.

a) Working on the entire real line $-\infty < x < \infty$, suppose that $f_0(x) = 2\sin(x) + 4\sin(2x)$ and $g_0(x) = 6\sin(x) - 4\sin(2x)$, where $f_0(x)$ and $g_0(x)$ are shorthand for $f(x,0)$ and $\dot{f}(x,0)$. Compute $f(x,t)$ for all x and t.

b) Now consider the same equation on the spatial interval $x \in [0,\pi]$ and impose the boundary conditions $f(0,t) = f(\pi,t) = 0$. In other words, we have a vibrating string of length $L = \pi$. Suppose that $f_0(x) = 2\sin(x) + 4\sin(2x)$ and $g_0(x) = 6\sin(x) - 4\sin(2x)$. What is $f(x,t)$?