M346 First Midterm Exam, July 25, 2011

1) (25 pts) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 8 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ (in $\mathbb{R}^3$). a) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

Row-reducing $\begin{pmatrix} 1 & 1 & 1 & 3 & | & 0 \\ 1 & 2 & 0 & 5 & | & 1 \\ 2 & 3 & 1 & 8 & | & 1 \end{pmatrix}$ yields $\begin{pmatrix} 1 & 0 & 2 & 1 & | & -1 \\ 0 & 1 & -1 & 2 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$. So

$$\begin{aligned} x_1 &= -1 & -2x_3 && -x_4 \\ x_2 &= 1 & +x_3 && -2x_4 \\ x_3 &= & x_3 && \\ x_4 &= & && x_4 \end{aligned}$$

or $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, where s and t are arbitrary numbers.

b) Find a basis for the column space of A

Row-reducing just A gives the same thing as before, only without the column after the bar, namely $A_{rref} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since there are pivots in the first two columns, a basis for $Col(A)$ is the first two columns of A, namely $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

c) Find a basis for the null space of A.

This also follows from A_{rref}, and is the vectors that are multiplied by s and t in part (a), namely $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

2. (15 points) Let $V = \mathbb{R}_2[t]$ be the space of quadratic polynomials in a variable t and consider the linear transformation $L(\mathbf{p}) = \mathbf{p}'(t) + \mathbf{p}(2t)$ from V to itself, where $\mathbf{p}'(t)$ is the derivative of $\mathbf{p}(t)$ and $\mathbf{p}(2t)$ means "plug in $2t$

instead of t". Find the matrix of this linear transformation with respect to the (standard) basis $\{1, t, t^2\}$.

Since $L(1) = 0 + 1 = 1$, $L(t) = 1 + 2t$, and $L(t^2) = 2t + (2t)^2 = 2t + 4t^2$, we have $[L]_{\mathcal{B}} = \begin{pmatrix} [1]_{\mathcal{B}} & [1+2t]_{\mathcal{B}} & [2t+4t^2]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$.

3. (25 pts) In $\mathbb{R}_2[t]$, let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis, and let $\mathcal{D} = \{1, 1+t, (1+t)^2\}$ be an alternate basis. Let $\mathbf{p}(t) = t^2 - 2t + 1$.

a) Compute the change-of-basis matrices $P_{\mathcal{BD}}$ and $P_{\mathcal{DB}}$.

$$P_{\mathcal{BB}} = \begin{pmatrix} [\mathbf{d}_1]_{\mathcal{B}} & [\mathbf{d}_2]_{\mathcal{B}} & [\mathbf{d}_3]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}. \; P_{\mathcal{DB}} = P_{\mathcal{BD}}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Use row reduction to take the inverse of $P_{\mathcal{BD}}$.)

b) Compute $[\mathbf{p}]_{\mathcal{D}}$. (Hint: first compute $[\mathbf{p}]_{\mathcal{B}}$.)

Since $[\mathbf{p}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $[\mathbf{p}]_{\mathcal{D}} = P_{\mathcal{DB}}[\mathbf{p}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$. You can also check that $t^2 - 2t + 1 = 4 - 4(t+1) + (t+1)^2$.

c) Let $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ be a linear operator such that $[L]_{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$.

Compute $[L]_{\mathcal{D}}$. (Remember that $[L]_{\mathcal{B}}$ is shorthand for $[L]_{\mathcal{BB}}$, and likewise for $[L]_{\mathcal{D}}$.)

$$[L]_{\mathcal{D}} = P_{\mathcal{DB}}[L]_{\mathcal{B}}P_{\mathcal{BD}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

4. (20 points) a) Find the characteristic polynomial of $\begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$. You do not need to find the eigenvalues or eigenvectors.

$$p_A(\lambda) = \det \begin{pmatrix} \lambda - 2 & -3 \\ -4 & \lambda \end{pmatrix} = \lambda^2 - 2\lambda - 12$$

b) Find the eigenvalues of $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. You do not have to find the eigenvectors.

5 and -1. The rows sum to 5, so 5 is an eigenvalue. The trace is 4, so the other eigenvalue is -1. You could also get this from the characteristic polynomial, or from the trace and determinant.

5. True or false? (15 points) You do not have to justify your answers, and partial credit will not be given.

a) If A is a 3×5 matrix and the column space of A is 2-dimensional, then the null space of A is also 2-dimensional.

FALSE. If the dimension of the column space is 2, then the dimension of the null space is $5 - 2 = 3$.

b) Every basis for $\mathbb{R}_2[t]$ contains exactly three vectors.

TRUE. That's what it means for $\mathbb{R}_2[t]$ to be 3-dimensional.

c) If L is a linear operator from a vector space V to itself, and if $\mathcal{B}$ and $\mathcal{D}$ are bases for V, then $[L]_\mathcal{D} = P_{\mathcal{DB}}[L]_\mathcal{B}$.

FALSE. The correct formula is $[L]_\mathcal{D} = P_{\mathcal{DB}}[L]_\mathcal{B} P_{\mathcal{BD}}$.

d) If $L : V \to V$ is a linear operator and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of V consisting of eigenvectors of L, then $[L]_\mathcal{B}$ is diagonal.

TRUE. Since $L(\mathbf{b}_i) = \lambda_i \mathbf{b}_i$, we have $[L(\mathbf{b}_i)]_\mathcal{B} = \lambda_i \mathbf{e}_i$, so $[L]_\mathcal{B} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

e) The eigenvalues of a real matrix are real.

FALSE. A real matrix has a real characteristic polynomial, but some of the roots of that polynomial may be complex. In particular, the eigenvalues of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ are $a \pm ib$.