M346 Second Midterm Exam Solutions, August 5, 2011

1) (25 pts) Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 8 & -2 \end{pmatrix}$.

a) Find the eigenvalues of A. For each eigenvalue, find a corresponding eigenvector.

The sum of each row is 6, the trace is 2, and the determinant is -24. From any two of these facts, or from the characteristic polynomial $\lambda^2 - 2\lambda - 24$, you get that the two eigenvalues are 6 and -4. From row-reducing $\lambda I - A$ you get that the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ (or $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} -1/4 \\ 1 \end{pmatrix}$). This means that $P = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$, so $P^{-1} = \frac{1}{-5}\begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$.

b) Compute A^n for all n. Make your answer as explicit as possible.

$A^n = PD^nP^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 6^n & 0 \\ 0 & (-4)^n \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$, which equals $\frac{1}{5}\begin{pmatrix} 4 \cdot 6^n + (-4)^n & 6^n - (-4)^n \\ 4 \cdot 6^n - 4(-4)^n & 6^n + 4(-4)^n \end{pmatrix}$.

c) Compute e^{At} as a function of t. Make your answer as explicit as possible.

$e^{At} = Pe^{Dt}P^{-1} = \frac{1}{5}\begin{pmatrix} 4e^{6t} + e^{-4t} & e^{6t} - e^{-4t} \\ 4e^{6t} - 4e^{-4t} & e^{6t} + 4e^{-4t} \end{pmatrix}$. This is the same answer as (b), only with e^{6t} instead of 6^n and with e^{-4t} instead of $(-4)^n$.

2. (30 points, 2 pages) Let $A = \begin{pmatrix} -4 & 5 \\ 5 & -4 \end{pmatrix}$. This matrix has eigenvalues 1 and -9 and corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find the solution to $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Since $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$. $\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so $\mathbf{y}(n) = \begin{pmatrix} 3 \cdot 1^n \\ (-9)^n \end{pmatrix}$, so $\mathbf{x}(n) = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-9)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 + (-9)^n \\ 3 - (-9)^n \end{pmatrix}$.

b) Find the solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Again $\mathbf{y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so $\mathbf{y}(t) = \begin{pmatrix} 3e^t \\ e^{-9t} \end{pmatrix}$ and $\mathbf{x}(t) = 3e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-9t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$

$\begin{pmatrix} 3e^t + e^{-9t} \\ 3e^t - e^{-9t} \end{pmatrix}$.

c) Find the solution to $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\dot{\mathbf{x}}(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Now we also have $\dot{\mathbf{y}}(0) = P^{-1}\dot{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Since the first eigenvalue is positive, $y_1(t) = y_1(0)\cosh(t) + \dot{y}_1(0)\sinh(t) = 3\cosh(t) + \sinh(t)$. The second eigenvalue is negative, so $y_2(t) = y_2(0)\cos(3t) + \dot{y}_2(0)\sin(3t)/3 = \cos(3t) + \sin(3t)$, so $\mathbf{x}(t) = \begin{pmatrix} 3\cosh(t) + \sinh(t) + \cos(3t) + \sin(3t) \\ 3\cosh(t) + \sinh(t) - \cos(3t) - \sin(3t) \end{pmatrix}$.

3. (15 pts) The differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= e^{-4x_1} - x_2 \\ \frac{dx_2}{dt} &= 5x_1x_2 \end{aligned}$$

have a fixed point at $x_1 = 0$, $x_2 = 1$. Find the linear approximation to these equations near $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and determine whether this fixed point is stable. Explain your reasoning!

Taking partial derivatives, we get that our matrix is $\begin{pmatrix} -4e^{-4x_1} & -1 \\ 5x_2 & 5x_1 \end{pmatrix}$ evaluated at $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} -4 & -1 \\ 5 & 0 \end{pmatrix}$. That is, we have $d\mathbf{y}/dt \approx \begin{pmatrix} -4 & -1 \\ 5 & 0 \end{pmatrix}\mathbf{y}$, where $\mathbf{y} = \mathbf{x} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The eigenvalues of the matrix are $-2 \pm i$. Since both eigenvalues have negative real part, both modes are stable, so the fixed point is stable.

4. (15 points)

On $\mathbb{R}_2[t]$ with the inner product $\langle \mathbf{f}|\mathbf{g}\rangle = \int_0^1 f(t)g(t)dt$, use Gram-Schmidt to convert $\{1, 2t, 6t^2\}$ to an orthogonal basis.

We start with $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = 2t$, $\mathbf{x}_3 = 6t^2$. We then take $\mathbf{y}_1 = \mathbf{x}_1 = 1$, $\mathbf{y}_2 = 2t - \frac{|1\rangle\langle 1|2t\rangle}{\langle 1|1\rangle} = 2t - \frac{1}{1}1 = 2t-1$. Then $\mathbf{x}_3 = 6t^2 - \frac{|1\rangle\langle 1|6t^2\rangle}{\langle 1|1\rangle} - \frac{2t-|1\rangle\langle 2t-1|6t^2\rangle}{\langle 2t-1|2t-1\rangle} = 6t^2 - \frac{2}{1}1 - \frac{1}{1/3}(2t-1) = 6t^2 - 3(2t-1) - 2 = 6t^2 - 6t + 1$.

5. (15 points) In $\mathbb{R}^5$ with the standard inner product, let V be the subspace spanned by $\begin{pmatrix}1\\2\\0\\1\\2\end{pmatrix}$ and $\begin{pmatrix}4\\3\\5\\-2\\1\end{pmatrix}$. Write $\mathbf{b} = \begin{pmatrix}7\\4\\5\\1\\2\end{pmatrix}$ as the sum of two vectors, one in V and the other orthogonal to V.

There are at least two ways to solve this.

Method 1: We are trying to write $\mathbf{b} = \mathbf{b}_\parallel + \mathbf{b}_\perp$, where $\mathbf{b}_\parallel$ is in V and $\mathbf{b}_\perp$ is orthogonal to V. However, a least-square solution to $A\mathbf{x} = \mathbf{b}$ is a true solution to $A\mathbf{x} = \mathbf{b}_\parallel$, so we just have to find a least-square solution, take $\mathbf{b}_\parallel = A\mathbf{x}$, and $\mathbf{b}_\perp = \mathbf{b} - \mathbf{b}_\parallel$. The matrix A is $\begin{pmatrix}1 & 4\\2 & 3\\0 & 5\\1 & -2\\2 & 1\end{pmatrix}$, and we get $A^TA = \begin{pmatrix}10 & 10\\10 & 55\end{pmatrix}$, while $A^T\mathbf{b} = \begin{pmatrix}20\\65\end{pmatrix}$. The least-squares solution is $\begin{pmatrix}1\\1\end{pmatrix}$, so $\mathbf{b}_\parallel = A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\5\\5\\-1\\3\end{pmatrix}$ and $\mathbf{b}_\perp = \begin{pmatrix}2\\-1\\0\\2\\-1\end{pmatrix}$.

Method 2: Use Gram-Schmidt to find an orthogonal basis for V. This yields $\mathbf{y}_1 = \begin{pmatrix}1\\2\\0\\1\\2\end{pmatrix}$ and $\mathbf{y}_2 = \mathbf{x}_2 - \frac{10}{10}\mathbf{y}_1 = \begin{pmatrix}3\\1\\5\\-3\\-1\end{pmatrix}$. Then $\mathbf{b}_\parallel = P_{\mathbf{y}_1}\mathbf{b} + P_{\mathbf{y}_2}\mathbf{b} = \frac{20}{10}\mathbf{y}_1 + \frac{45}{45}\mathbf{y}_2$ and $\mathbf{b}_\perp = \mathbf{b} - \mathbf{b}_\parallel$, which gives the same answer as method 1.