M346 First Midterm Exam, July 25, 2011

1) (25 pts) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 8 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ (in $\mathbb{R}^3$). a) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

b) Find a basis for the column space of A

c) Find a basis for the null space of A.

2. (15 points) Let $V = \mathbb{R}_2[t]$ be the space of quadratic polynomials in a variable t and consider the linear transformation $L(\mathbf{p}) = \mathbf{p}'(t) + \mathbf{p}(2t)$ from V to itself, where $\mathbf{p}'(t)$ is the derivative of $\mathbf{p}(t)$ and $\mathbf{p}(2t)$ means "plug in $2t$ instead of t". Find the matrix of this linear transformation with respect to the (standard) basis $\{1, t, t^2\}$.

3. (25 pts) In $\mathbb{R}_2[t]$, let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis, and let $\mathcal{D} = \{1, 1+t, (1+t)^2\}$ be an alternate basis. Let $\mathbf{p}(t) = t^2 - 2t + 1$.

a) Compute the change-of-basis matrices $P_{\mathcal{BD}}$ and $P_{\mathcal{DB}}$.

b) Compute $[\mathbf{p}]_{\mathcal{D}}$. (Hint: first compute $[\mathbf{p}]_{\mathcal{B}}$.)

c) Let $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ be a linear operator such that $[L]_{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. Compute $[L]_{\mathcal{D}}$. (Remember that $[L]_{\mathcal{B}}$ is shorthand for $[L]_{\mathcal{BB}}$, and likewise for $[L]_{\mathcal{D}}$.)

4. (20 points) a) Find the characteristic polynomial of $\begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$. You do not need to find the eigenvalues or eigenvectors.

b) Find the eigenvalues of $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. You do not have to find the eigenvectors.

5. True or false? (15 points) You do not have to justify your answers, and partial credit will not be given.

a) If A is a 3×5 matrix and the column space of A is 2-dimensional, then the null space of A is also 2-dimensional.

b) Every basis for $\mathbb{R}_2[t]$ contains exactly three vectors.

c) If L is a linear operator from a vector space V to itself, and if $\mathcal{B}$ and $\mathcal{D}$ are bases for V, then $[L]_{\mathcal{D}} = P_{\mathcal{DB}}[L]_{\mathcal{B}}$.

d) If $L : V \to V$ is a linear operator and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of V consisting of eigenvectors of L, then $[L]_{\mathcal{B}}$ is diagonal.

e) The eigenvalues of a real matrix are real.