Artin fans

Simons symposium on Non-Archimedean and Tropical Geometry

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Superabundance

Mikhalkin-Speyer: there is a tropical cubic curve C of genus 1 in TP^3 which does not lift to an algebraic curve (Speyer, *Tropical Geometry*, Berkeley thesis 2005, Figure 5.1).

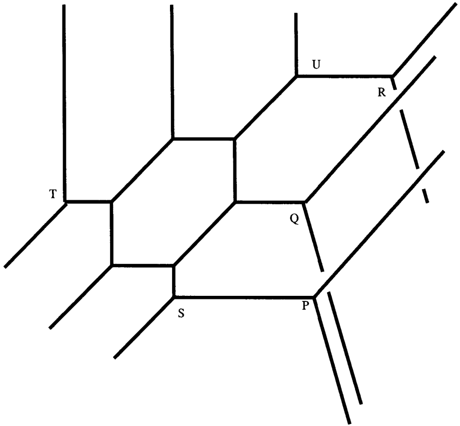


Figure 5.1: A Genus 1 Zero Tension Curve which is not Tropical

Superabundance (continued)

I want to understand this phenomenon.

Principles:

- Tropical curves $\sqsubset \to TP^3$ encode in detail degenerations of curves $C \to \mathbb{P}^3$
- They encode **logarithmic stable maps** $C \to \mathbb{P}^3$.
- superabundance $\Longleftrightarrow$ obstructedness
- I wish to describe a fairy-tale world in which this issue disappears, and is useful for geometers.

Logarithmic structures (Kato, Fontaine, Illusie)

Definition

A *pre logarithmic structure* is

$$X = (\underline{X}, M \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}}) \qquad \text{or just} \quad (\underline{X}, M)$$

such that

- $\underline{X}$ is a scheme - the *underlying scheme*
- M is a sheaf of monoids on X, and
- α is a monoid homomorphism, where the monoid structure on $\mathcal{O}_{\underline{X}}$ is the multiplicative structure.

Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}^*_{\underline{X}} \to \mathcal{O}^*_{\underline{X}}$ is an isomorphism.

Examples

Examples

- $(\underline{X}, \mathcal{O}^*_{\underline{X}} \hookrightarrow \mathcal{O}_{\underline{X}})$, the trivial *logarithmic* structure.
- Let $\underline{X}, D \subset \underline{X}$ be a variety with a divisor. We define $M_D \hookrightarrow \mathcal{O}_{\underline{X}}$:

$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \;\middle|\; f_{U \smallsetminus D} \in \mathcal{O}^\times_{\underline{X}}(U \smallsetminus D) \right\}.$$

- Let k be a field, $\underline{X} = \operatorname{Spec} k$, define the *punctured point*:

$$\begin{array}{ccc} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

The magic of logarithmic geomery

Definition

Logarithmic smoothness = loc.fin. type + local lifting property.

Theorem (Kato)

$f : X \to Y$ is log smooth if étale locally it is the pullback of a toric morphism: locally on Y

$$\begin{array}{ccccc} X & \xrightarrow{\text{étale}} & Y \times_{\operatorname{Spec} R[M]} \operatorname{Spec} R[N] & \longrightarrow & \operatorname{Spec} R[N] \\ & & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec} R[M] \end{array}$$

for a reasonable monoid homomorphism $M \to N$

- Any toric or toroidal variety X is logarithmically smooth over $\operatorname{Spec} k$.

$$T_X \simeq \mathcal{O}_{\underline{X}}^{\dim \underline{X}}.$$

- A nodal curve is logarithmically smooth over a punctured point.

Here be monsters!

Logarithmic obstructions to deforming a logarithmic map $C \to \mathbb{P}^3$ lie in sequence

$$H^1(\underline{C}, T_{\underline{C}}) \to H^1(\underline{C}, \mathcal{O}^3_{\underline{C}}) \to Obs \to 0.$$

These can be nonzero on a broken cubic curve! The example of Mikhalkin - Speyer is such.

Artin fans

Olsson:

$$\{\text{Logarithmic structures } X \text{ on } \underline{X}\} \quad \longleftrightarrow \quad \{\underline{X} \to \underline{\mathrm{Log}}\}.$$

The stack Log *is huge and does not specify combinatorial data.*

Theorem (Wise; ℵ, Chen, Marcus)

There is an initial factorization $X \to \mathcal{A}_X \to \mathrm{Log}$ such that $\mathcal{A}_X \to \mathrm{Log}$ is étale, representable, strict.

The stack $\mathcal{A}_X$ is small, totally combinatorial.
The requirement "representable" is a compromise.
Example: $\mathcal{A}_{\mathbb{A}^1} = [\mathbb{A}^1/\mathbb{G}_m]$. In general for toric X, $\mathcal{A}_X = [X/T]$.

$\mathbb{P}^3$ and $\mathcal{A}_{\mathbb{P}^3}$

$$\mathbb{P}^3 = (\mathbb{A}^4 \setminus \{0\})/\mathbb{G}_m.$$

So

$$\{C \to \mathbb{P}^3\} \leftrightarrow \{(\mathcal{L}, s_0, \ldots, s_3) | s_i \text{ do not vanish together}\}.$$

Now

$$\mathcal{A}_{\mathbb{P}^3} = (\mathbb{A}^4 \setminus \{0\})/\mathbb{G}_m^4.$$

So

$$\{C \to \mathcal{A}_{\mathbb{P}^3}\} \leftrightarrow \{((\mathcal{L}_0, s_0), \ldots, (\mathcal{L}_3, s_3)) | s_i \text{ do not vanish together}\}.$$

The monsters evaporate!

$$T_{\mathbb{P}^3} = \mathcal{O}^3, \text{ but } T_{\mathcal{A}_{\mathbb{P}^3}} = 0.$$

Logarithmic obstructions to deforming a logarithmic map $C \to \mathcal{A}_{\mathbb{P}^3}$ lie in a quotient of

$$H^1(\underline{C}, 0) = 0.$$

The obstructions are gone!

Sample theorem

Theorem ($\aleph$-Wise)

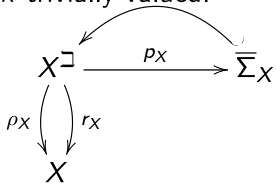
If $Y \to X$ is a toroidal modification, then

> *Logarithmic Gromov–Witten invariants of X coincide with those of Y.*

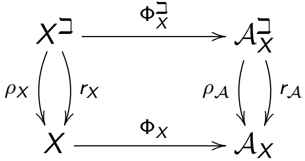
Reason: $\mathfrak{M}(\mathcal{A}_Y) \to \mathfrak{M}(\mathcal{A}_X)$ is birational. So $\overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X)$ is virtually birational.

Nonarchimedean picture

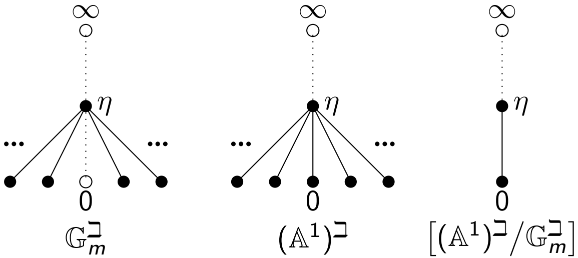
X - log smooth over $k = \bar{k}$ trivially valued.
Thuillier introduced:



here $X^{\beth}$ is the Berkovich analytic formal fiber and $\overline{\Sigma}_X$ its skeleton / extended cone complex.
Ulirsch introduced an analytification of $X \to \mathcal{A}_X$:



I claim that $\mathcal{A}_X^{\beth}$ is familiar to the audience:



So $\mathcal{A}_{\mathbb{A}^1}^{\beth}$ is homeomorphic to $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$, the skeleton of $\mathbb{A}^1$.

In general we have a homeomorphism $\mathcal{A}_X^{\beth} \sim \overline{\Sigma}_X$. The complete diagram is

$$\begin{array}{ccccc} X^{\beth} & \longrightarrow & \mathcal{A}_X^{\beth} & \longrightarrow & \overline{\Sigma}_X \\ {\scriptstyle \rho_X} \downarrow \downarrow {\scriptstyle r_X} & & {\scriptstyle \rho_{\mathcal{A}}} \downarrow \downarrow {\scriptstyle r_{\mathcal{A}}} & & {\scriptstyle \rho_{\Sigma}} \downarrow \downarrow {\scriptstyle r_{\Sigma}} \\ X & \longrightarrow & \mathcal{A}_X & \longrightarrow & F_X \end{array}$$

Here, at least when X has Zariski charts,

- F_X is the underlying monoidal space of $\mathcal{A}_X$, the Kato fan of X.
- The complex $\overline{\Sigma}_X$ can be identified as $F_X(\mathbb{R}_{\geq 0} \sqcup \{\infty\})$