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Asymptotic cohomological functions of toric divisors

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Abstract

We study functions on the class group of a toric variety measuring the rates of growth of the cohomology groups of multiples of divisors. We show that these functions are piecewise polynomial with respect to finite polyhedral chamber decompositions. As applications, we express the self-intersection number of a T-Cartier divisor as a linear combination of the volumes of the bounded regions in the corresponding hyperplane arrangement and prove an asymptotic converse to Serre vanishing.

Keywords: Toric varieties; Linear systems; Asymptotic invariants; Gelfand–Kapranov–Zelevinsky decompositions; Fans

1. Introduction

Suppose D is an ample divisor on an n-dimensional algebraic variety. The sheaf cohomology of $\mathcal{O}(D)$ does not necessarily reflect the positivity of D; $\mathcal{O}(D)$ may have few global sections and its higher cohomology groups may not vanish. However, for $m \gg 0$, $\mathcal{O}(mD)$ is globally generated and all of its higher cohomology groups vanish. Moreover, the rate of growth of the space of global sections of $\mathcal{O}(mD)$ as m increases carries information on the positivity of D.

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Indeed, if we write $h^0(mD)$ for the dimension of $H^0(X, \mathcal{O}(mD))$, then by asymptotic Riemann–Roch [11, Example 1.2.19],

$$(D^n) = \lim_m \frac{h^0(mD)}{m^n/n!}.$$

In general, when D is not necessarily ample, this limit exists and is called the volume of D. It is written $\hat{h}^0(D)$ or $\mathrm{vol}(D)$. The regularity of the rate of growth of the cohomology groups of $\mathcal{O}(mD)$ for $m \gg 0$ contrasts with the subtlety of the behavior of the cohomology of $\mathcal{O}(D)$ itself and motivates the study of asymptotic cohomological functions of divisors.

Lazarsfeld has shown that the volume of a Cartier divisor depends only on its numerical equivalence class and that the volume function extends to a continuous function on $N^1(X)_{\mathbb{R}}$ [11, 2.2.C]. The volume function is polynomial on the ample cone, where it agrees with the top intersection form. In some special cases, including for toric varieties, smooth projective surfaces, abelian varieties, and generalized flag varieties, the volume function is piecewise polynomial with respect to a locally finite polyhedral chamber decomposition of the interior of the effective cone. The behavior of the volume function outside the ample cone is known to be more complicated in general [2]. In this paper, we study the volume function and its generalizations, the higher asymptotic cohomological functions, in the toric case.

Let $X = X(\Delta)$ be a complete n-dimensional toric variety. Let D be a T-Weil[3] divisor on X, and P_D the associated polytope in $M_{\mathbb{R}}$. Since $h^0(mD)$ is the number of lattice points in P_{mD}, and since $P_{mD} = mP_D$ for all positive integers m, $\hat{h}^0(D)$ is the volume of P_D, normalized so that the smallest lattice simplex in $M_{\mathbb{R}}$ has unit volume. Oda and Park describe, in combinatorial language, a finite polyhedral chamber decomposition of the effective cone in the divisor class group of a toric variety, which they call the Gelfand–Kapranov–Zelevinsky (or GKZ) decomposition, such that the combinatorial structure of P_D is constant as D varies within each chamber [12]. It follows that $\mathrm{vol}\, P_D$ is polynomial on each of these chambers; see [1, VIII.5 Problem 10] and [4, Example 2.8]. In particular, $\hat{h}^0$ extends to a continuous, piecewise polynomial function with respect to a finite polyhedral decomposition of the effective cone in $A_{n-1}(X)_{\mathbb{R}}$. The piecewise polynomial behavior of $\hat{h}^0$ also follows from [3, Proposition 4.13], since toric varieties have "finitely generated linear series." The GKZ decomposition also arises as the decomposition of the effective cone into "Mori chambers" and "variation of GIT chambers," see [7]. In an appendix, we give a brief, self-contained account of GKZ decompositions in the language of toric divisors.

Generalizing the volume function, we define higher asymptotic cohomological functions of toric divisors by

$$\hat{h}^i(D) = \lim_m \frac{h^i(mD)}{m^n/n!},$$

where $h^i(mD)$ is the dimension of $H^i(X, \mathcal{O}(mD))$.[4] In the toric case, it follows from local cohomology computations of Eisenbud, Mustaţă, and Stillman [5] that there is a decomposi-

[3] Asymptotic cohomological functions of Cartier divisors on general (not necessarily toric) varieties have been studied in [2,10] and [11, 2.2.C]. In this paper, we allow Weil divisors wherever possible. Our approach is self-contained, and does not rely on results from the general theory.

[4] The second author has studied higher asymptotic cohomological functions of line bundles on general varieties, defined similarly but with a lim sup instead of a limit, and has shown that these extend to continuous functions on $N^1(X)_{\mathbb{R}}$ [10]. It is not known whether the limits exist in general.

tion of $M_{\mathbb{R}}$ into finitely many polyhedral regions such that the dimensions of the graded pieces $H^i(X, \mathcal{O}(D))_u$ are constant for lattice points u in each region. The regions are indexed by collections of rays $I \subset \Delta(1)$, and for $D = \sum d_\rho D_\rho$, they are given by

$$P_{D,I} := \left\{ u \in M_{\mathbb{R}} \colon \langle u, v_\rho \rangle \geqslant -d_\rho \text{ if and only if } \rho \in I \right\}.$$

In particular, the regions for mD are the m-fold dilations of the regions for D. In Section 3, we deduce from this that the limit in the definition of $\hat{h}^i$ exists, and that each $\hat{h}^i$ extends to a continuous, piecewise polynomial function with respect to a finite polyhedral decomposition of $A_{n-1}(X)_{\mathbb{R}}$. Tchoudjem has given a similar combinatorial description of the cohomology of equivariant line bundles on regular compactifications of reductive groups [13, Théorème 2.1]; it should be interesting to investigate whether the results and techniques of this paper extend to such varieties.

We apply our cohomology computations to give a formula for the self-intersection number of a T-Cartier divisor. For each $I \subset \Delta(1)$, let Δ_I be the fan consisting of exactly those cones in Δ spanned by rays in I, and let $\Delta_I(j)$ be the set of j-dimensional cones in Δ_I. Define

$$\chi(\Delta_I) := \sum_{j=0}^{n} (-1)^j \cdot \#\Delta_I(j).$$

In Section 2, we show that

$$\chi(\mathcal{O}(D)) = (-1)^n \sum_{P_{D,I} \text{ bounded}} \chi(\Delta_I) \cdot \#(P_{D,I} \cap M).$$

Using this formula for $\chi(\mathcal{O}(D))$ and asymptotic Riemann–Roch, we give a self-intersection formula for T-Cartier divisors. When $P_{D,I}$ is bounded, we write vol $P_{D,I}$ for the volume of $P_{D,I}$, normalized so that the smallest lattice simplex has unit volume.

Theorem 1 *(Self-intersection formula). Let X be a complete n-dimensional toric variety and D a T-Cartier divisor on X. Then*

$$\left(D^n\right) = (-1)^n \cdot \sum_{P_{D,I} \text{ bounded}} \chi(\Delta_I) \cdot \operatorname{vol} P_{D,I}.$$

When X is smooth, Theorem 1 is closely related to a formula of Karshon and Tolman for the pushforward of the top exterior power of a presymplectic form under the moment map [8]. In this case, the coefficient $(-1)^n \cdot \chi(\Delta_I)$ is equal to a winding number which gives the density of the Duistermaat–Heckman measure on $P_{D,I}$.

We conclude by proving an "asymptotic converse" to Serre vanishing in the toric case. From Serre vanishing we know that, for D ample, $h^i(mD) = 0$ for all $i > 0$ and $m \gg 0$. The set of ample divisors is open in $\operatorname{Pic}(X)_{\mathbb{R}}$, so the higher volume functions vanish in a neighborhood of every ample divisor. We prove the converse for divisors on complete simplicial toric varieties.

Theorem 2 *(Asymptotic converse to Serre vanishing). Let D be a divisor on a complete simplicial toric variety. Then D is ample if and only if $\hat{h}^i$ vanishes identically in a neighborhood of D in* $\operatorname{Pic}(X)_{\mathbb{R}}$ *for all $i > 0$.*

The asymptotic converse to Serre vanishing does not hold in general if X is complete but not simplicial. Fulton gives an example of a complete, nonprojective toric threefold with no nontrivial line bundles [6, pp. 25–26, 72]. For such an X, $\mathrm{Pic}(X) = 0$ and all of the $\hat{h}^i$ vanish, but the zero divisor is not ample. We do not know whether the asymptotic converse to Serre vanishing holds for nonsimplicial projective toric varieties.

On a toric variety, linear equivalence and numerical equivalence of Cartier divisors coincide, so $\mathrm{Pic}(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$. Lazarsfeld asks whether, for a smooth complex projective variety X, a divisor D is ample if and only if the higher asymptotic cohomological functions vanish in a neighborhood of the class of D in $N^1(X)_{\mathbb{R}}$.[5]

2. Cohomology of T-Weil divisors

By the cohomology groups of a Weil divisor D on an algebraic variety X, we always mean the sheaf cohomology groups $H^i(X, \mathcal{O}(D))$, where $\mathcal{O}(D)$ is the sheaf whose sections over U are the rational functions f such that $(\operatorname{div} f + D)|_U$ is effective. When X is complete, we write $h^i(D)$ for the dimension of $H^i(X, \mathcal{O}(D))$.

In this section, for each T-Weil divisor D on a toric variety, we give a decomposition of the weight space $M_{\mathbb{R}}$ into finitely many polyhedral regions such that the dimension of the u-graded piece of the ith cohomology group of D is constant for all u in each region. This decomposition can be deduced from local cohomology computations in [5, Theorem 2.7], but we present a proof using different methods. Our approach is a variation on the standard method for computing the cohomology groups of T-Cartier divisors [6, Section 3.5].

Let $X = X(\Delta)$ be an n-dimensional toric variety over a field k, and let $\Delta(1)$ be the set of rays of Δ. Let $D = \sum d_\rho D_\rho$ be a T-Weil divisor. For each $I \subset \Delta(1)$, define

$$P_{D,I} := \{u \in M_{\mathbb{R}}\colon\ \langle u, v_\rho\rangle \geqslant -d_\rho \text{ if and only if } \rho \in I\},$$

and let Δ_I be the subfan of Δ consisting of exactly those cones whose rays are contained in I. Note that $P_{D,\Delta(1)}$ is the closed polyhedron usually denoted P_D, each $P_{D,I}$ is a polyhedral region defined by an intersection of halfspaces, some closed and some open, and $M_{\mathbb{R}}$ is their disjoint union. With D fixed, for each $u \in M$, set

$$I_u := \{\rho \in \Delta(1)\colon\ \langle u, v_\rho\rangle \geqslant -d_\rho\}.$$

Recall that $H^i_{|\Delta_I|}(|\Delta|)$ denotes the topological local cohomology group of $|\Delta|$ with support in $|\Delta_I|$. Here and throughout, all topological homology and cohomology groups are taken with coefficients in k, the base field of X.

Proposition 1. *Let $X = X(\Delta)$ be a toric variety, D a T-Weil divisor on X. Then*

$$H^i(X, \mathcal{O}(D)) \cong \bigoplus_{u \in M} H^i_{|\Delta_{I_u}|}(|\Delta|).$$

[5] In the time since this article was written, de Fernex, Lazarsfeld, and the second author have announced an affirmative answer to this question [4, Theorem 4.7].

Proof. The Čech complex $C^\bullet$ that computes the cohomology of $\mathcal{O}(D)$ is M-graded, and the u-graded piece is a direct sum of u-graded pieces of modules of sections of $\mathcal{O}(D)$ as follows:

$$C_u^i = \bigoplus_{\sigma_0,\ldots,\sigma_i \in \Delta} H^0\big(U_{\sigma_0} \cap \cdots \cap U_{\sigma_i}, \mathcal{O}(D)\big)_u.$$

Now $H^0(U_{\sigma_0} \cap \cdots \cap U_{\sigma_i}, \mathcal{O}(D))_u$ is isomorphic to k if $\sigma_0 \cap \cdots \cap \sigma_i$ is in Δ_{I_u}, and is zero otherwise. In particular,

$$H^0\big(U_{\sigma_0} \cap \cdots \cap U_{\sigma_i}, \mathcal{O}(D)\big)_u \cong H^0_{|\Delta_{I_u}| \cap \sigma_0 \cap \cdots \cap \sigma_i}(\sigma_0 \cap \cdots \cap \sigma_i).$$

A standard argument from topology [6, Lemma p.75] shows that the Čech complex $C^\bullet$ also computes $H^i_{|\Delta_{I_u}|}(|\Delta|)$. $\square$

Corollary 1. *Let $X = X(\Delta)$ be a toric variety, D a T-Weil divisor on X. Then*

$$H^i\big(X, \mathcal{O}(D)\big) \cong \bigoplus_{I \subset \Delta(1)} \left(\bigoplus_{u \in P_{D,I} \cap M} H^i_{|\Delta_I|}\big(|\Delta|\big) \right).$$

Proof. Since $P_{D,I} \cap M$ is exactly the set of u such that $I_u = I$, the corollary follows from Proposition 1 by regrouping the summands. $\square$

Proposition 2. *Let $X = X(\Delta)$ be a complete toric variety. For D a T-Weil divisor on X,*

$$h^i(D) = \sum_{P_{D,I}\ \text{bounded}} h^i_{|\Delta_I|}(N_{\mathbb{R}}) \cdot \#(P_{D,I} \cap M).$$

Proof. When X is complete, the support of Δ is all of $N_{\mathbb{R}}$, and $H^i(X, \mathcal{O}(D))$ is finite dimensional. By Corollary 1, $H^i_{|\Delta_I|}(|\Delta|)$ must vanish whenever $P_{D,I}$ is unbounded, and the result follows. $\square$

If S is the unit sphere for some choice of coordinates on $N_{\mathbb{R}}$, then by [6, Exercise p. 88],

$$h^i_{|\Delta_I|}(N_{\mathbb{R}}) \cong \widetilde{h}_{n-i-1}\big(|\Delta_I| \cap S\big).$$

Therefore, Proposition 2 implies that computations of cohomology groups of toric divisors can be reduced to computations of reduced homology groups of finite polyhedral cell complexes and counting lattice points in polytopes. We will use the following lemma to show that the reduced homology computations are not necessary if one is only interested in the Euler characteristic $\chi(\mathcal{O}(D))$. For any fan Σ, let $\Sigma(j)$ denote the set of j-dimensional cones in Σ, and define

$$\chi(\Sigma) := \sum_{j=0}^{n} (-1)^j \cdot \#\Sigma(j).$$

Lemma 1. *Let Σ be a fan in $N_{\mathbb{R}}$. Then*

$$\sum_{i=0}^{n}(-1)^i \cdot h^i_{|\Sigma|}(N_{\mathbb{R}}) = (-1)^n \cdot \chi(\Sigma).$$

Proof. Let S be the unit sphere for some choice of coordinates on $N_{\mathbb{R}}$. Then

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i \cdot h^i_{|\Sigma|}(N_{\mathbb{R}}) &= \sum_{i=0}^{n}(-1)^i \cdot \tilde{h}_{n-i-1}\big(|\Sigma| \cap S\big) \\ &= (-1)^n + \sum_{i=0}^{n}(-1)^i \cdot h_{n-i-1}\big(|\Sigma| \cap S\big).\end{aligned}$$

Setting $j = n - i$, and then using the correspondence between the $(j-1)$-dimensional cells in $|\Sigma| \cap S$ and the j-dimensional cones in Σ, we have

$$\begin{aligned}(-1)^n + \sum_{i=0}^{n}(-1)^i \cdot h_{n-i-1}\big(|\Sigma| \cap S\big) &= (-1)^n + \sum_{j=0}^{n}(-1)^{n-j} \cdot h_{j-1}\big(|\Sigma| \cap S\big) \\ &= (-1)^n \cdot \sum_{j=0}^{n}(-1)^j \cdot \#\Sigma(j). \qquad \square\end{aligned}$$

Proposition 3. *Let D be a T-Weil divisor on a complete n-dimensional toric variety. Then*

$$\chi\big(\mathcal{O}(D)\big) = (-1)^n \cdot \sum_{P_{D,I} \text{ bounded}} \chi(\Delta_I) \cdot \#(P_{D,I} \cap M).$$

Proof. The proposition follows immediately from Proposition 2 and Lemma 1. $\square$

3. Asymptotic cohomological functions and the self-intersection formula

Definition 1. Let X be a complete n-dimensional toric variety. The ith asymptotic cohomological function $\hat{h}^i : A_{n-1}(X) \to \mathbb{R}$ is defined by

$$\hat{h}^i(D) = \lim_m \frac{h^i(mD)}{m^n/n!}.$$

For a bounded polyhedral region $P \subset M_{\mathbb{R}}$, let $\operatorname{vol} P$ denote the volume of P, normalized so that the smallest lattice simplex has unit volume. Note that

$$\operatorname{vol} P = \lim_m \frac{\#mP \cap M}{m^n/n!}.$$

Proposition 4. *Let D be a T-Weil divisor on a complete toric variety $X = X(\Delta)$. Then*

$$\hat{h}^i(D) = \sum_{P_{D,I} \text{ bounded}} h^i_{|\Delta_I|}(N_{\mathbb{R}}) \cdot \operatorname{vol} P_{D,I}.$$

Proof. For all $I \subset \Delta(1)$, and for all positive integers m, $P_{mD,I} = m P_{D,I}$. The proposition therefore follows immediately from the definition of $\hat{h}^i$ and Proposition 2. $\square$

Corollary 2. *Let X be a complete n-dimensional toric variety. Then $\hat{h}^i$ extends to a continuous, piecewise polynomial function with respect to a finite polyhedral decomposition of $A_{n-1}(X)_{\mathbb{R}}$.*

Proof. The set of I such that $P_{D,I}$ is bounded does not depend on D. Indeed, $P_{D,I}$ is bounded if and only if there is no hyperplane in $N_{\mathbb{R}}$ separating the rays in I from the rays in $\Delta(1) \setminus I$. The result then follows from Proposition 4 since, for each such I, $\operatorname{vol} P_{D,I}$ extends to a continuous, piecewise polynomial function with respect to a finite polyhedral decomposition of $A_{n-1}(X)_{\mathbb{R}}$. $\square$

Theorem 1 *(Self-intersection formula). Let X be a complete n-dimensional toric variety and D a T-Cartier divisor on X. Then*

$$\left(D^n\right) = (-1)^n \cdot \sum_{P_{D,I} \text{ bounded}} \chi(\Delta_I) \cdot \operatorname{vol} P_{D,I}.$$

Proof. By asymptotic Riemann–Roch [9, VI.2], when D is Cartier,

$$\left(D^n\right) = \lim_m \frac{\chi(\mathcal{O}(mD))}{m^n/n!}.$$

The theorem then follows from Proposition 3. $\square$

4. An asymptotic converse to Serre vanishing

We begin by briefly recalling the Gelfand–Kapranov–Zelevinsky (GKZ) decomposition introduced by Oda and Park [12] and a few of its basic properties. Assume that X is complete. The GKZ decomposition is a fan whose support is the effective cone in $A_{n-1}(X)_{\mathbb{R}}$ and whose maximal cones are in 1–1 correspondence with the simplicial fans Σ in $N_{\mathbb{R}}$ such that $\Sigma(1) \subset \Delta(1)$ and $X(\Sigma)$ is projective. We call the interior of a maximal GKZ cone a *GKZ chamber*, and write γ_Σ for the GKZ chamber corresponding to Σ. If D is a T-Weil divisor whose class $[D]$ lies in γ_Σ, then Σ is the normal fan to P_D. This property fully characterizes the GKZ decomposition. We will need the following basic property relating divisors in γ_Σ to divisors on $X(\Sigma)$: if f denotes the birational map from X to $X(\Sigma)$ induced by the identity on N, then the birational transform $f_*(D)$ is ample on $X(\Sigma)$, and $P_{f_*(D)} = P_D$. See Appendix A for proofs and for a more detailed discussion of the GKZ decomposition in the language of toric divisors.

Lemma 2. *Let γ_Σ be a GKZ chamber, and let f be the birational map from $X = X(\Delta)$ to $X(\Sigma)$ induced by the identity on N. Let $D_1, \dots, D_r$ be distinct prime T-invariant divisors on X corresponding to rays $\rho_1, \dots, \rho_r \in \Delta$, respectively. For D a T-Weil divisor with $[D] \in \gamma_\Sigma$,*

$$\frac{\partial^r \hat{h}^0}{\partial D_1 \cdots \partial D_r}(D) = \frac{n!}{(n-r)!} \cdot \big(f_*(D)^{n-r} \cdot f_*(D_1) \cdot \cdots \cdot f_*(D_r)\big).$$

In particular, $\frac{\partial^r \hat{h}^0}{\partial D_1 \cdots \partial D_r}$ is strictly positive on γ_Σ if $\rho_1, \dots, \rho_r$ span a cone in Σ and vanishes identically on γ_Σ otherwise.

Proof. Suppose $r = 1$. Since $f_*(D)$ is ample and $P_{f_*(D)} = P_D$ for D in γ_Σ, $\hat{h}^0$ is given on γ_Σ by $D \mapsto (f_*(D)^n)$. Therefore,

$$\begin{aligned}\frac{\partial \hat{h}^0}{\partial D_1} &= \lim_{\epsilon \to 0} \frac{(f_*(D + \epsilon D_1)^n) - (f_*(D)^n)}{\epsilon} \\ &= n\big(f_*(D)^{n-1} \cdot f_*(D_1)\big).\end{aligned}$$

The general case follows by a similar computation and induction on r. The last statement follows from the formula, since $f_*(D)$ is ample and $f_*(D_1) \cdot \cdots \cdot f_*(D_r)$ is an effective cycle if $\rho_1, \dots, \rho_r$ span a cone in Σ and is zero otherwise [6, Chapter 5]. □

Theorem 2 *(Asymptotic converse to Serre vanishing). Let D be a divisor on a complete simplicial toric variety. Then D is ample if and only if $\hat{h}^i$ vanishes identically in a neighborhood of D in* $\mathrm{Pic}(X)_\mathbb{R}$ *for all $i > 0$.*

Proof. Since the limits in the definition of the $\hat{h}^i$ exist, by asymptotic Riemann–Roch, for D a $\mathbb{Q}$-Cartier divisor, $(D^n) = \sum_{i=0}^n (-1)^i \cdot \hat{h}^i(D)$. Therefore, if $\hat{h}^i$ vanishes in a neighborhood of D for all $i > 0$, then $\hat{h}^0$ agrees with the top intersection form in a neighborhood of D. To prove Theorem 2, we will prove the stronger fact that if $\hat{h}^0$ agrees with the top intersection form in a neighborhood of D, then D is ample. It will suffice to show that if γ_Σ is a GKZ chamber and $\hat{h}^0(D) = (D^n)$ for $[D] \in \gamma_\Sigma$, then $\Sigma = \Delta$.

Suppose γ_Σ is a GKZ chamber and $\hat{h}^0(D) = (D^n)$ for $[D] \in \gamma_\Sigma$. Let $\rho_1, \dots, \rho_n$ be rays spanning a maximal cone $\sigma \in \Delta$. It will suffice to show that $\rho_1, \dots, \rho_n$ span a cone in Σ. On γ_Σ, since $\hat{h}^0$ agrees with the top intersection form,

$$\begin{aligned}\frac{\partial^n \hat{h}^0}{\partial D_1 \cdots \partial D_n} &= n! \cdot (D_1 \cdot \cdots \cdot D_n) \\ &= n! \cdot \mathrm{mult}(\sigma)^{-1}.\end{aligned}$$

In particular, $\frac{\partial^n \hat{h}^0}{\partial D_1 \cdots \partial D_n}$ does not vanish identically on γ_Σ. By Lemma 2, $\rho_1, \dots, \rho_n$ span a cone in Σ, as required. □

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Appendix A. Gelfand–Kapranov–Zelevinsky decompositions

In this appendix we give a self-contained account of the GKZ decompositions of Oda and Park [12], in the language of toric divisors.

A possibly degenerate fan in N is a finite collection Σ of convex (not necessarily strongly convex) rational polyhedral cones in $N_{\mathbb{R}}$ such that every face of a cone in Σ is in Σ, and the intersection of any two cones in Σ is a face of each. The intersection of all of the cones in Σ is the unique linear subspace $L_\Sigma \subset N_{\mathbb{R}}$ that is a face of every cone in Σ; we say that Σ is degenerate if L_Σ is not zero. Associated to Σ is a toric variety X_Σ of dimension $\dim N_{\mathbb{R}} - \dim L_\Sigma$, whose torus is $T_{N/L_\Sigma \cap N}$, and the T-Cartier divisors on X_Σ correspond naturally and bijectively to the piecewise linear functions on $|\Sigma|$ whose restriction to L_Σ is identically zero.

Let $X = X(\Delta)$ be an n-dimensional toric variety and assume, for simplicity, that $|\Delta|$ is convex and n-dimensional. Let $D = \sum d_\rho D_\rho$ be an effective T-$\mathbb{Q}$-Weil divisor and let $P_D = \{u \in M\colon \langle u, v_\rho\rangle \geqslant -d_\rho\}$ be the polyhedron associated to D. From P_D one constructs the possibly degenerate normal fan Σ_D, whose support is $|\Delta|$ and whose cones are in one to one order reversing correspondence with the faces of P_D; the cone corresponding to a face Q is

$$\sigma_Q = \bigl\{v \in |\Delta|\colon \langle u, v\rangle \geqslant \langle u', v\rangle \text{ for all } u \in P_D \text{ and } u' \in Q\bigr\}.$$

Note that σ_Q is positively spanned by those rays $\rho \in \Delta(1)$ such that the affine hyperplane $\langle u, v_\rho\rangle = -d_\rho$ contains Q.

We define a convex piecewise linear function Ξ_D on $|\Delta|$ by

$$\Xi_D(v) = \min\bigl\{\langle u, v\rangle\colon u \in P_D\bigr\}.$$

The maximal cones of Σ_D are the maximal domains of linearity of Ξ_D. When D is $\mathbb{Q}$-Cartier and ample, $\Sigma_D = \Delta$ and $\Xi_D = \Psi_D$ is the piecewise linear function usually associated to D [6, Section 3.3]. It follows from the definition of Ξ_D that

$$\Xi_D(v_\rho) \geqslant -d_\rho, \tag{1}$$

with equality for those ρ such that the affine hyperplane $\langle u, v_\rho\rangle = -d_\rho$ contains a face of P_D. Let $I_D \subset \Delta(1)$ be the set of rays for which equality does not hold in (1).

Definition 2 *(GKZ cones).* Let Σ be a possibly degenerate fan whose support is $|\Delta|$, such that X_Σ is quasiprojective, and such that there is a set of rays $I \subset \Delta(1)$ for which every cone in Σ is positively spanned by rays in $\Delta(1) \smallsetminus I$. The GKZ cone $\gamma_{\Sigma,I}$ is defined to be

$$\gamma_{\Sigma,I} := \bigl\{[D] \in A_{n-1}(X)_{\mathbb{Q}}\colon \Sigma \text{ refines } \Sigma_D \text{ and } I_D \subseteq I\bigr\}.$$

The GKZ cone $\gamma_{\Sigma,I}$ is well defined since Σ_D and I_D depend only on the linear equivalence class of D.

GKZ Decomposition Theorem. [12, Theorem 3.5] *The GKZ cone $\gamma_{\Sigma,I}$ is a rational polyhedral cone of dimension* $\dim \operatorname{Pic}(X_\Sigma)_{\mathbb{Q}} + \#I$. *The set of GKZ cones is a fan whose support is the effective cone in $A_{n-1}(X)_{\mathbb{R}}$, and the faces of $\gamma_{\Sigma,I}$ are exactly those $\gamma_{\Sigma',I'}$ such that Σ refines Σ' and $I' \subset I$.*

It follows from the theorem that the maximal GKZ cones are in 1–1 correspondence with the nondegenerate simplicial fans Σ in $N_{\mathbb{R}}$ such that $\Sigma(1) \subset \Delta(1)$, $|\Sigma| = |\Delta|$, and $X(\Sigma)$ is quasiprojective. Indeed, if $[D]$ is in $\gamma_{\Sigma,I}$ then $\dim P_D = n - \dim L_\Sigma$. In particular, if Σ is degenerate then $\gamma_{\Sigma,I}$ is in the boundary of the effective cone and $\hat{h}^0|_{\gamma_{\Sigma,I}}$ is identically zero. If Σ is nondegenerate, then

$$\dim \gamma_{\Sigma,I} \leqslant \#\Sigma(1) - n + \#I \leqslant \#\Delta(1) - n,$$

with equalities if and only if Σ is simplicial and $I = \Delta(1) \setminus \Sigma(1)$. The interiors of the maximal cones of the GKZ decomposition are called GKZ chambers, and we write γ_Σ for the GKZ chamber corresponding to Σ.

In order to prove the GKZ Decomposition Theorem, we need a few basic tools relating divisors on X to divisors on X_Σ, where Σ is a possibly degenerate fan in $N_{\mathbb{R}}$ whose support is $|\Delta|$. Let ϕ_Σ be the map taking a T-$\mathbb{Q}$-Cartier divisor D on X_Σ to the T-$\mathbb{Q}$-Weil divisor $\phi_\Sigma(D)$ on X, where

$$\phi_\Sigma(D) = \sum_{\rho \in \Delta(1)} -\Psi_D(v_\rho) D_\rho.$$

Note that ϕ_Σ respects linear equivalence and induces an injection of $\operatorname{Pic}(X_\Sigma)$ into $A_{n-1}(X)$. The map ϕ_Σ may be realized geometrically as follows. Let $\widetilde{X}$ be the toric variety corresponding to the smallest common refinement of Δ and Σ. The identity on N induces morphisms p_1 and p_2 from $\widetilde{X}$ to X and to X_Σ, respectively. Then $\phi_\Sigma = {p_1}_* \circ p_2^*$.

The following lemma will be used to prove the GKZ Decomposition Theorem. It also shows that $\gamma_{\Sigma,I}$ is equal to the cone $\operatorname{cpl}(\Sigma, \Delta(1) \setminus I)$ defined in [12, Section 3].

Lemma 3. *Let $D = \sum d_\rho D_\rho$ be a T-$\mathbb{Q}$-Weil divisor. The following are equivalent*:

(i) *The GKZ cone $\gamma_{\Sigma,I}$ contains $[D]$.*
(ii) *There is a convex function Ξ that is linear on each maximal cone of Σ such that $\Xi(v_\rho) \geqslant -d_\rho$ for all $\rho \in \Delta(1)$, with equality when $\rho \notin I$.*
(iii) *There is a divisor $\widetilde{D}$ linearly equivalent to D and a decomposition $\widetilde{D} = \phi_\Sigma(D') + E$ such that D' is a nef T-$\mathbb{Q}$-Cartier divisor on X_Σ, $P_{\widetilde{D}} = P_{D'}$, and E is an effective divisor whose support is contained in $\bigcup_{\rho \in I} D_\rho$.*

Proof. If $[D]$ is in $\gamma_{\Sigma,I}$, then (ii) holds for $\Xi = \Xi_D$. If (ii) holds, then choose $u \in M_{\mathbb{Q}}$ such that $\Xi|_{L_\Sigma} = u|_{L_\Sigma}$. Let $\widetilde{D} = D + \sum_\rho \langle u, v_\rho \rangle D_\rho$, and let D' be the $\mathbb{Q}$-Cartier divisor on X_Σ corresponding to $\Xi - u$. Since Ξ is convex, D' is nef, and (iii) holds with

$$E = \sum_\rho \bigl(\Xi(v_\rho) + d_\rho\bigr) D_\rho.$$

Also, since $\Psi_{D'} = \Xi_D - u = \Xi_{\widetilde{D}}$, we have $P_{D'} = P_{\widetilde{D}}$.

It remains to show that (iii) implies (i). Replacing D by $\widetilde{D}$ if necessary, we may assume $D = \phi_\Sigma(D') + E$, where D' is a nef $\mathbb{Q}$-Cartier divisor on X_Σ and E is an effective divisor whose support is contained in $\bigcup_{\rho \in I} D_\rho$. We must show that Σ refines Σ_D and $\Xi_D(v_\rho) = -d_\rho$ for $\rho \notin I$. Since $P_D = P_{D'}$, and since D' is nef, $\Xi_D = \Psi_{D'}$, which is linear on each cone of Σ.

Hence Σ_D is refined by Σ. Since the support of E is contained in $\bigcup_{\rho\in I} D_\rho$, we also have $\Xi_D(v_\rho) = -d_\rho$ for $\rho \notin I$, as required. $\square$

Corollary 3. *Let D be a T-Weil divisor on X. Then $[D]$ is in the relative interior of $\gamma_{\Sigma,I}$ if and only if $\Sigma_D = \Sigma$ and $I_D = I$.*

Proof. The decomposition in Lemma 3, part (iii) is essentially unique; if $\widetilde{D}$ is replaced by $\widetilde{D} + \sum \langle u, v_\rho\rangle D_\rho$, where $u|_{L_\Sigma} = 0$, then D' is replaced by the divisor corresponding to $\Psi_{D'} - u$ and $E = \sum_{\rho\in I} e_\rho D_\rho$ remains fixed. It follows that the map $[D] \mapsto (D', (e_\rho)_{\rho\in I})$ gives an isomorphism

$$\gamma_{\Sigma,I} \xrightarrow{\sim} \mathrm{Nef}(X_\Sigma) \times \mathbb{R}^I_{\geqslant 0}.$$

Taking relative interiors gives $\gamma^\circ_{\Sigma,I} \xrightarrow{\sim} \mathrm{Ample}(X_\Sigma) \times \mathbb{R}^I_{>0}$. Therefore $[D]$ is in the relative interior of $\gamma_{\Sigma,I}$ if and only if Ξ_D is strictly convex with respect to Σ and the inequality in (1) is strict exactly when $\rho \in I$. $\square$

Corollary 4. *Suppose X is complete, and let $\gamma_{\Sigma,I}$ be a GKZ cone, with Σ nondegenerate. Let f be the birational map from X to X_Σ induced by the identity on N. If $[D] \in \gamma_{\Sigma,I}$ (respectively $[D] \in \gamma^\circ_{\Sigma,I}$), then $P_{f_*(D)} = P_D$ and $f_*(D)$ is nef (respectively $f_*(D)$ is ample). In particular, $\hat{h}^0|_{\gamma_{\Sigma,I}}$ is given by $[D] \mapsto (f_*(D)^n)$.*

Proof. Since Σ is nondegenerate, we can take $\widetilde{D} = D$ and let $D = \phi_\Sigma(D') + E$ be the decomposition in Lemma 3, part (iii). Then, since $P_D = P_{D'}$, $\hat{h}^0(D) = \hat{h}^0(D')$. Furthermore, since D' is nef on X_Σ (and is ample if $[D] \in \gamma^\circ_{\Sigma,I}$), $\hat{h}^0(D') = ((D')^n)$. We claim that $D' = f_*(D)$. Indeed,

$$D' = \sum_{\rho\in\Sigma(1)} -\Xi_D(v_\rho) f_*(D_\rho) = f_*(D),$$

and the result follows. $\square$

Corollary 5. *The volume function of a complete toric variety is given by distinct polynomials on distinct GKZ chambers.*

Proof. Let $\gamma_\Sigma, \gamma_{\Sigma'}$ be distinct GKZ chambers. Let $\rho_1, \ldots, \rho_n$ be rays spanning a maximal cone in Σ that is not in Σ'. By Lemma 2, $\frac{\partial^n \hat{h}^0}{\partial D_1 \cdots \partial D_n}$ vanishes identically on $\gamma_{\Sigma'}$, but not on γ_Σ. $\square$

Proof of the GKZ Decomposition Theorem. First, we claim that $[D]$ is in $\gamma_{\Sigma,I}$ if and only if for each maximal cone $\sigma \in \Sigma$, for each collection of linearly independent rays $\rho_1, \ldots, \rho_n$ in $\Delta(1) \setminus I$ that are contained in σ, and for each $\rho \in \Delta(1)$ with $v_\rho = a_1 v_{\rho_1} + \cdots + a_n v_{\rho_n}$, we have

$$-a_1 d_{\rho_1} - \cdots - a_n d_{\rho_n} \begin{cases} = -d_\rho, & \text{if } \rho \subset \sigma \text{ and } \rho \notin I, \\ \geqslant -d_\rho, & \text{otherwise.} \end{cases} \tag{2}$$

There are only finitely many such conditions, and all of the coefficients are rational, so the claim implies that each $\gamma_{\Sigma,I}$ is a convex rational polyhedral cone. Suppose (2) holds. Let $u_\sigma \in M_\mathbb{Q}$ be such that $\langle u_\sigma, v_{\rho_i}\rangle = -d_{\rho_i}$ for $1 \leqslant i \leqslant n$. The equalities in (2) ensure that the u_σ glue together to give a continuous piecewise linear function Ξ on $|\Sigma|$, where $\Xi|_\sigma = u_\sigma$, such that $\Xi(v_\rho) = -d_\rho$ for $\rho \notin I$. The inequalities in (2) guarantee that Ξ is convex. By part (ii) of Lemma 3, it follows that $[D]$ is in $\gamma_{\Sigma,I}$. Conversely, if $[D]$ is in $\gamma_{\Sigma,I}$, then we have a Ξ as in part (ii) of Lemma 3. Say $\Xi|_\sigma = u_\sigma$. Then the left-hand side of (2) is equal to $\langle u_\sigma, v_\rho\rangle$ and the desired equalities and inequalities follow from the choice of Ξ.

It follows from Corollary 3 that the effective cone in $A_{n-1}(X)_{\mathbb{R}}$ is the disjoint union of the relative interiors of the GKZ cones, and that $\dim \gamma_{\Sigma,I} = \dim \operatorname{Pic}(X_\Sigma)_{\mathbb{Q}} + \#I$. Any finite collection of rational polyhedral cones such that every face of a cone in the collection is in the collection, and such that the relative interiors of the cones are disjoint, is a fan. The faces of a cone in such a collection are precisely the cones in the collection that it contains. Therefore, to prove the theorem, it remains only to show that every face of a GKZ cone is a GKZ cone. Let $\gamma_{\Sigma,I}$ be a GKZ cone, let $\rho_1, \ldots, \rho_n$ be linearly independent rays contained in a maximal cone $\sigma \in \Sigma$, let $\rho \in \Delta(1)$ with $v_\rho = a_1 v_{\rho_1} + \cdots + a_n v_{\rho_n}$, and let $\tau \preccurlyeq \gamma_{\Sigma,I}$ be the face where equality holds in (2). If $\rho \subset \sigma$, then $\tau = \gamma_{\Sigma, I\smallsetminus\{\rho\}}$. If $\rho \not\subset \sigma$ then consider the set of convex cones σ' in $\mathbb{N}_{\mathbb{R}}$ which are unions of maximal cones in Σ, which contain σ and ρ, and are such that $X(\Sigma')$ is quasiprojective, where Σ' is the fan whose maximal cones are σ' and all of the maximal cones of Σ that are not contained in σ'. This set is nonempty since it contains $|\Sigma|$, and since it is closed under intersections it must contain a minimal element $\bar{\sigma}$. Let $\overline{\Sigma}$ be the corresponding fan. Then $\tau = \gamma_{\overline{\Sigma},I}$. □

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