Tropical curves and top weight cohomology of $M_g$

BMS Colloquium

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The moduli space of curves

\[ M_g := \text{moduli space of smooth projective curves of genus } g \]

- Quotient of Teichmüller space by Mod\( (S_g) \)
- Smooth orbifold of complex dimension \( 3g - 3 \)
- Central to algebraic geometry, topology, string theory, etc.

- Origins in the work of Riemann
The moduli space of curves

Why study rational cohomology $H^*(M_g)$?

- Group cohomology of $\text{Mod}(S_g)$
- Characteristic classes of families of curves
- Intricate combinatorial structure
- 0% understood

And because it’s there
Boundary complexes

\[ X = \text{smooth algebraic variety over } \mathbb{C} \]
\[ \overline{X} = \text{simple normal crossings compactification} \]
\[ \partial \overline{X} := \overline{X} \setminus X \text{ union of smooth transverse divisors } D_1, \ldots, D_s \]

\[ \Delta(\partial \overline{X}) = \text{dual complex} \]

vertices \( v_1, \ldots, v_s \quad \leftrightarrow \quad \text{irred. comps of } \partial \overline{X} \)

edges \[ [v_i, v_j] \quad \leftrightarrow \quad \text{irred. comps of } D_i \cap D_j \]

2-faces \[ \langle v_i, v_j, v_k \rangle \quad \leftrightarrow \quad \text{irred. comps of } D_i \cap D_j \cap D_k \]

\ldots

u. s. w.
Examples

\[
\begin{align*}
X_1 &= \mathbb{C}^n, \quad \bar{X}_1 = \mathbb{P}^n, \quad \Delta(\partial \bar{X}_1) = \text{pt.} \\
X_2 &= (\mathbb{C}^\times)^n, \quad \bar{X}_2 = \mathbb{P}^n, \quad \Delta(\partial \bar{X}_2) = \partial \Delta^n
\end{align*}
\]
Top weight cohomology

Let \( d = \dim(X) \).

**Theorem (Deligne 1971)**

There is a natural isomorphism

\[
\tilde{H}_{k-1}(\Delta(\partial \overline{X})) \cong \Gr_{2d}^{W} H^{2d-k}(X).
\]

**Example:**

\[
H_{n-1}(\partial \Delta^n) \cong H^n((\mathbb{C}^\times)^n)
\]

**Theorem (Danilov 1975, Thuillier 2007, P 2013)**

The simple homotopy type of \( \Delta(\partial \overline{X}) \) depends only on \( X \).
Boundary of the moduli space

\( \overline{M}_g = \) moduli space of stable curves (Deligne and Mumford 1969)
\( \partial \overline{M}_g = \) divisor with normal crossings

**Theorem (Abramovich, Caporaso, and P 2015)**

The dual complex \( \Delta_g \) of \( \partial \overline{M}_g \) is the moduli space of stable tropical curves of genus \( g \) and volume 1.

\( \Delta_g \) has other natural interpretations as:
- The quotient of the curve complex by the action of \( \text{Mod}(S_g) \)
- The quotient of the simplicial completion of Outer Space by the action of \( \text{Out}(F_g) \)
Example: $g = 2$

Poset of strata in $\partial \overline{M}_2$

$\Delta_2$
Fundamental group and low genus computations

\( H^*(M_g) \) is known for \( g \leq 4 \):

**Theorem (Looijenga 1993 and Tommasi 2005)**

1. \( \Delta_3 \) is a rational homology 5-sphere;
2. \( \Delta_4 \) has the rational homology of a point.

**Theorem (Allcock, Corey and P 2019)**

1. \( \pi_1(\Delta_g) = 1 \) for all \( g \);
2. \( \Delta_3 \) is homotopy equivalent to \( S^5 \);
3. \( \Delta_4 \) is not contractible.

There is 3-torsion in \( H_5(\Delta_4; \mathbb{Z}) \) and 2-torsion in \( H_6 \) and \( H_7 \).
The rational cohomology of $M_g$

The low degree cohomology of $M_g$ stabilizes:

**Theorem (Madsen and Weiss 2007)**

$$H^\ast(M_g) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots], \text{ for } * \leq \frac{2}{3}(g - 1).$$

And the high degree cohomology vanishes:

**Theorem (Harer 1984)**

$$\text{vcd}(M_g) = 4g - 5.$$

**Theorem (Church-Farb-Putman, Morita-Sakasai-Suzuki 2013)**

$$H^{4g-5}(M_g) = 0 \text{ for all } g.$$
Tautological ring and Euler characteristic

\[ R^*(M_g) := \text{subring generated by the } \kappa_i \]

- Supported in even degrees up to \(2g - 4\),
- Intricately structured (Faber, Graber, Pandharipande, Petersen, Pixton, Tommasi, Vakil, ... (1997– )
- Growth is subexponential:
  \[ \dim_{\mathbb{Q}} R^*(M_g) < C\sqrt{g}. \]

The full cohomology ring grows much faster:

**Theorem (Harer and Zagier 1986)**

\[ (-1)^{g+1} \chi(M_g) \sim g^{2g} \]

As of 1986, there was not a single odd cohomology group \(H^k(M_g)\) that was known to be nonzero(!)
The first odd cohomology

Theorem (Tommasi 2005)

\[ H^5(M_4) \cong \mathbb{Q} \]
Rational homology of $\Delta_g$

Rational cellular chain complex $C_{k-1}(\Delta_g)$ gen. by isomorphism classes of pairs $(G, \omega)$:

- $G$ is a dual graph of a stable curve of genus $g$ with $k$ nodes
- $\omega$ is a total ordering of the $k$-edges
- with relations:

\[(G, \omega) = \text{sgn}(\sigma)(G, \sigma\omega), \text{ for } \sigma \in S_k.\]

- and differential:

\[d(G, \omega) = \sum_i (-1)^i (G/e_i, \omega_i).\]
Example

\[ K_4 \]

- \([K_4] \neq 0 \) in \( C_5(\Delta_3) \)
- \( d([K_4]) = 0 \), and its class spans \( H_5(\Delta_3) \cong \mathbb{Q} \)

Claim: \([K_n] \in \text{Im}(d)\) for \( n > 4 \).

Sketch of proof: The vanishing of \( H^*(M_g) \) for \( g > 4g - 6 \) implies that \( C_*(\Delta_g) \) is acyclic in degrees less than \( 2g - 1 \).
We note that $W_g$ and $W_0^g$ represent cells of degree $2g - 2$ and $g$, respectively.

When $g$ is even, $W_g$ and $W_0^g$ have automorphisms that act by odd permutations on the edges, and hence are zero as cellular chains.

When $g$ is odd, these graphs do not have automorphisms that act by odd permutations on the edges, and Lemma A.1 implies immediately that $W_g$ and $W_0^g$ represent rational cycles on $g$ and $g - 1$, respectively. Moreover, they are nonzero.

Lemma A.3. For $g > 3$ odd, $W_g$ and $W_0^g$ represent nonzero homology classes in $H_2(g - 1; \mathbb{Q})$ and $H_2(g - 1; \mathbb{Q})$, respectively.

Proof. (We suppress signs and orientations throughout.) The fact that $W_g$ represents a nontrivial class is established by [Wil15]; see [CGP18, Theorem 2.6]. As for $W_0^g$, applying the transfer homomorphism gives $t(W_g) = \pm (g^2 - g)$ and $W_0^g = \pm gW_0^{g-1}$.

Theorem 1.7 and Lemma A.2 imply that $W_0^g$ (and $W_0^{g-1}$) are nontrivial.

Corollary

$$H_{14}(\Delta_6) \cong \mathbb{Q}$$

$$H_{15}(M_6) \neq 0.$$
More examples

- $d([W_g]) = 0$
- $[W_g] \neq 0$ for $g$ odd
- Is $[W_g]$ in $\text{Im}(d)$?
- No for $g = 3, 5$
- No for $g = 7$ (computer calculation)
Theorem

\([W_g] \not\in \text{Im}(d) \text{ for } g \text{ odd.}\)

Idea of proof: The class \([W_g]\) is nonzero in \(H_{2g-1}(\Delta_g)\) if it pairs nontrivially with some cocycle.

Following Drinfeld and Willwacher:

- \(\alpha_G := \) Feynman amplitude in 3d Chern-Simons theory
- Express as an integral over a configuration space
- Structured so that \(\sigma_g = \sum_G \alpha[G]\) is a cocycle
- Compute, using tricks from analytic number theory:

\[\alpha_{W_g} \in \mathbb{Q}^\times \zeta(g)\]
Growth of \( H^{4g-6}(M_g) \)

Nonvanishing of \([W_g]\) implies \( H^{4g-6}(M_g) \neq 0 \) for \( g \) odd

Theorem (Chan, Galatius and P 2018)

\[
\dim_{\mathbb{Q}} H^{4g-6}(M_g) > 1.3247^g + C.
\]
Sketch of proof

- Subcomplex $\Delta_{lw} \subset \Delta_g$ parametrizing tropical curves with loop edges or vertices of positive weight is contractible
- $C_*(\Delta_g, \Delta_{lw})$ is isomorphic to the graph complex $K(g)$
- $\prod_g H^*(\Delta_g)$ carries a Lie bracket, additive in $g$ and $\#E(G)$

Theorem (Willwacher 2015)

$$\prod_g (H^{2g-1}(\Delta_g)) \cong \text{grt}_1.$$

Theorem (Brown 2012)

$$\text{grt}_1 \supset \text{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots).$$

$$\dim_{\mathbb{Q}} \text{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots)_g \gg \beta^g,$$

for $\beta < \beta_0 \approx 1.3247\ldots$ real root of $t^3 - t - 1$.  

\[\Box\]
Unexpected

Conjecture (Kontsevich 1993, Church-Farb-Putman 2014)

\[ H^{4g-4-k}(M_g) = 0, \quad \text{for } g \gg k. \]

- Kontsevich stated six parallel conjectures, for the associative, commutative, and lie operads, with even and odd coefficients.
- This is the “associative, odd” conjecture.
- The “commutative, even” conjecture follows from vanishing of \( H_k(\Delta_g) \) for \( k < 2g - 1 \).
Further applications

**Theorem (CGP 2018; Khoroshkin-Willwacher-Živković 2017)**

The following cohomology groups do not vanish:

\[ H^{21}(M_8), \, H^{23}(M_8), \, H^{27}(M_9), \, H^{27}(M_{10}), \, H^{31}(M_{10}). \]

**Theorem (CGP 2018; Khoroshkin-Willwacher-Živković 2017)**

The odd top weight cohomology grows exponentially:

\[ \dim_{\mathbb{Q}} \bigoplus_{g' < g} \text{Gr}^{W}_{6g'-6} H^{\text{odd}}(M_{g'}) > 1.3247^g + C. \]
That’s all folks

Vielen Dank!