Tropical curves and top weight cohomology of M_g

BMS Colloquium

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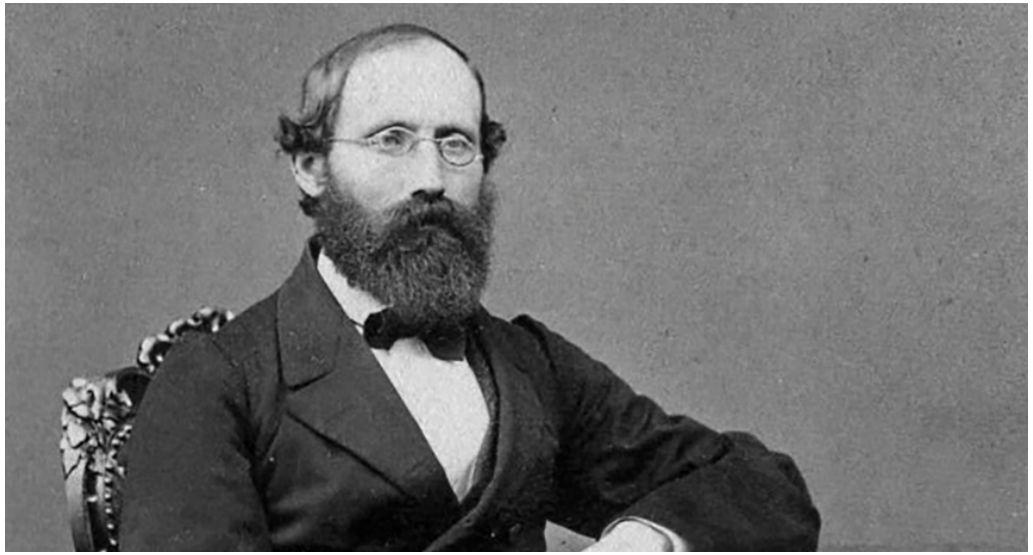
The moduli space of curves

$M_g :=$ moduli space of smooth projective curves of genus g

- Quotient of Teichmüller space by $\mathrm{Mod}(S_g)$
- Smooth orbifold of complex dimension $3g - 3$
- Central to algebraic geometry, topology, string theory, etc.

- Origins in the work of Riemann



The moduli space of curves

Why study rational cohomology $H^*(M_g)$?

- Group cohomology of $\mathrm{Mod}(S_g)$
- Characteristic classes of families of curves
- Intricate combinatorial structure
- 0% understood

And because it's there

Boundary complexes

X = smooth algebraic variety over $\mathbb{C}$
$\overline{X}$ = simple normal crossings compactification
$\partial\overline{X} := \overline{X} \smallsetminus X$ union of smooth transverse divisors $D_1, \ldots, D_s$

$\Delta(\partial\overline{X})$ = dual complex

vertices $v_1, \ldots, v_s$	$\longleftrightarrow$	irred. comps of $\partial\overline{X}$
edges $[v_i, v_j]$	$\longleftrightarrow$	irred. comps of $D_i \cap D_j$
2-faces $\langle v_i, v_j, v_k \rangle$	$\longleftrightarrow$	irred. comps of $D_i \cap D_j \cap D_k$
	$\ldots$	
	u. s. w.	

Examples

$$\boxed{X_1 = \mathbb{C}^n, \qquad \overline{X}_1 = \mathbb{P}^n, \qquad \Delta(\partial\overline{X}_1) = \mathrm{pt}.}$$

$$\boxed{X_2 = (\mathbb{C}^\times)^n, \qquad \overline{X}_2 = \mathbb{P}^n, \qquad \Delta(\partial\overline{X}_2) = \partial\Delta^n}$$

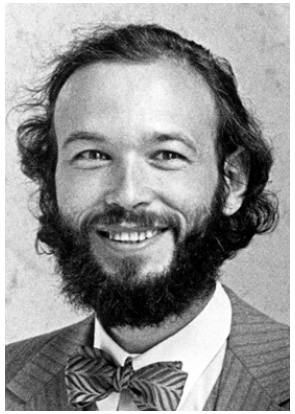
Top weight cohomology

Let $d = \dim(X)$.

Theorem (Deligne 1971)

There is a natural isomorphism

$$\widetilde{H}_{k-1}(\Delta(\partial\overline{X})) \cong \mathrm{Gr}^W_{2d} H^{2d-k}(X).$$



Example: $H_{n-1}(\partial\Delta^n) \cong H^n((\mathbb{C}^\times)^n)$

Theorem (Danilov 1975, Thuillier 2007, P 2013)

The simple homotopy type of $\Delta(\partial\overline{X})$ depends only on X.

Boundary of the moduli space

$\overline{M}_g$ = moduli space of stable curves (Deligne and Mumford 1969)

$\partial\overline{M}_g$ = divisor with normal crossings

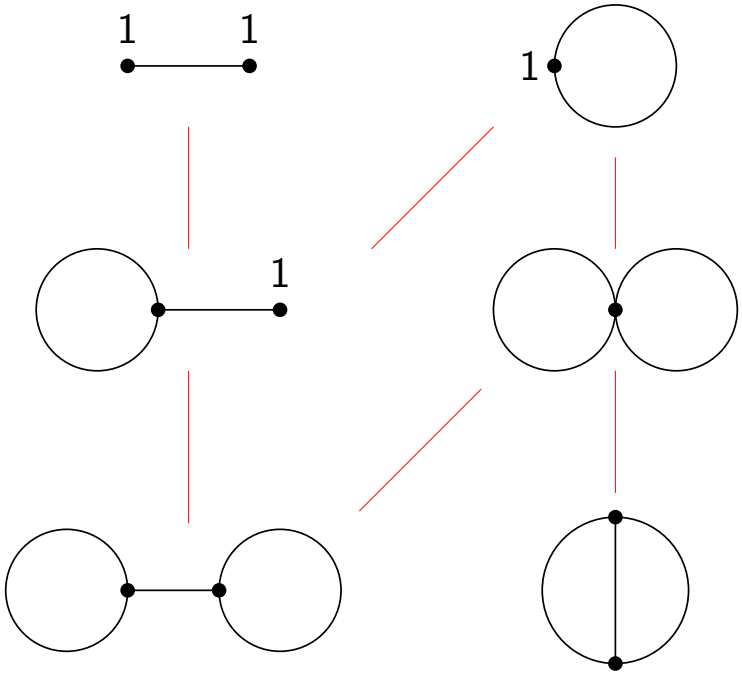
Theorem (Abramovich, Caporaso, and P 2015)

The dual complex Δ_g of $\partial\overline{M}_g$ is the moduli space of stable tropical curves of genus g and volume 1.

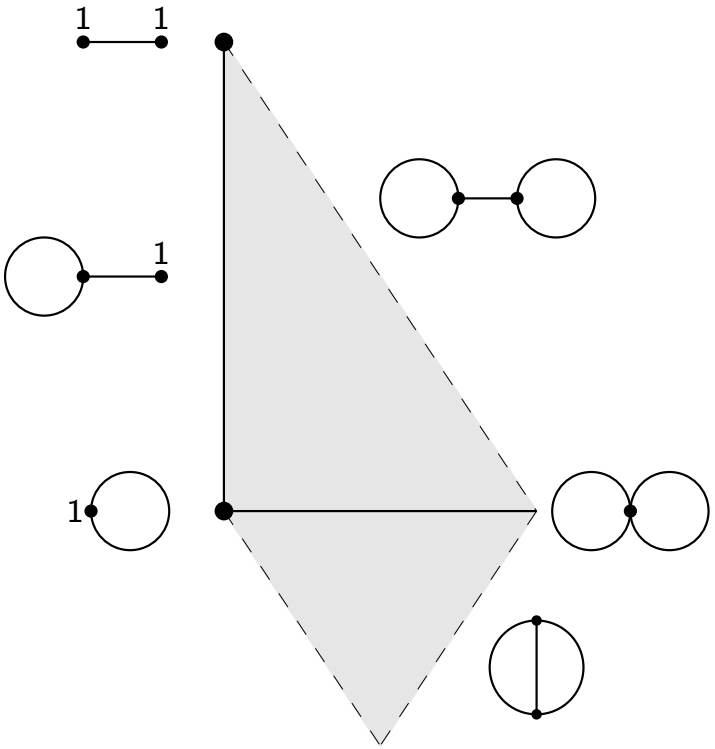
Δ_g has other natural interpretations as:

- The quotient of the curve complex by the action of $\mathrm{Mod}(S_g)$
- The quotient of the simplicial completion of Outer Space by the action of $\mathrm{Out}(\mathrm{F_g})$

Example: $g = 2$



Poset of strata in $\partial\overline{M}_2$



Δ_2

Fundamental group and low genus computations

$H^*(M_g)$ is known for $g \leq 4$:

Theorem (Looijenga 1993 and Tommasi 2005)

1. Δ_3 *is a rational homology* 5*-sphere;*
2. Δ_4 *has the rational homology of a point.*

Theorem (Allcock, Corey and P 2019)

1. $\pi_1(\Delta_g) = 1$ *for all* g*;*
2. Δ_3 *is homotopy equivalent to* S^5*;*
3. Δ_4 *is not contractible.*

There is 3-torsion in $H_5(\Delta_4; \mathbb{Z})$ and 2-torsion in H_6 and H_7

The rational cohomology of M_g

The low degree cohomology of M_g stabilizes:

Theorem (Madsen and Weiss 2007)

$$H^*(M_g) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots], \textit{ for } * \leq \tfrac{2}{3}(g-1).$$

And the high degree cohomology vanishes:

Theorem (Harer 1984)

$$\mathrm{vcd}(M_g) = 4g - 5.$$

Theorem (Church-Farb-Putman, Morita-Sakasai-Suzuki 2013)

$$H^{4g-5}(M_g) = 0 \textit{ for all } g.$$

Tautological ring and Euler characteristic

$R^*(M_g) :=$ subring generated by the κ_i

- Supported in even degrees up to $2g-4$,
- Intricately structured (Faber, Graber, Pandharipande, Petersen, Pixton, Tommasi, Vakil, ... (1997–))
- Growth is subexponential:

$$\dim_{\mathbb{Q}} R^*(M_g) < C^{\sqrt{g}}.$$

The full cohomology ring grows much faster:

Theorem (Harer and Zagier 1986)

$$(-1)^{g+1}\chi(M_g) \sim g^{2g}$$

As of 1986, there was not a single odd cohomology group $H^k(M_g)$ that was known to be nonzero(!)

The first odd cohomology

Theorem (Tommasi 2005)

$$H^5(M_4) \cong \mathbb{Q}$$

Rational homology of Δ_g

Rational cellular chain complex $C_{k-1}(\Delta_g)$ gen. by isomorphism classes of pairs (G, ω):

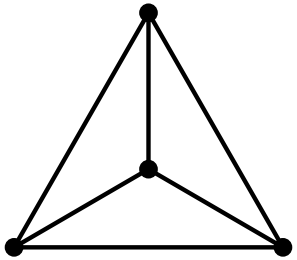
- G is a dual graph of a stable curve of genus g with k nodes
- ω is a total ordering of the k-edges
- with relations:

$$(G, \omega) = \mathrm{sgn}(\sigma)(G, \sigma\omega), \text{ for } \sigma \in S_k.$$

- and differential:

$$d(G, \omega) = \sum_i (-1)^i (G/e_i, \omega_{\hat{i}}).$$

Example



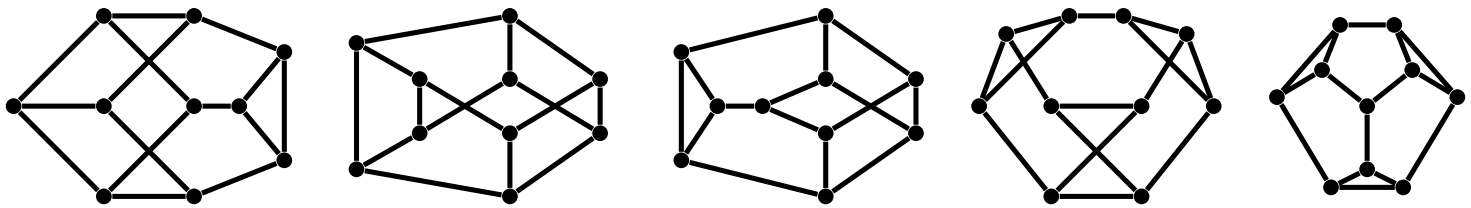
K_4

- $[K_4] \neq 0$ in $C_5(\Delta_3)$
- $d([K_4]) = 0$, and its class spans $H_5(\Delta_3) \cong \mathbb{Q}$

Claim: $[K_n] \in \mathrm{Im}(d)$ for $n > 4$.

Sketch of proof: The vanishing of $H^*(M_g)$ for $g > 4g - 6$ implies that $C_*(\Delta_g)$ is acyclic in degrees less than $2g - 1$.

More examples



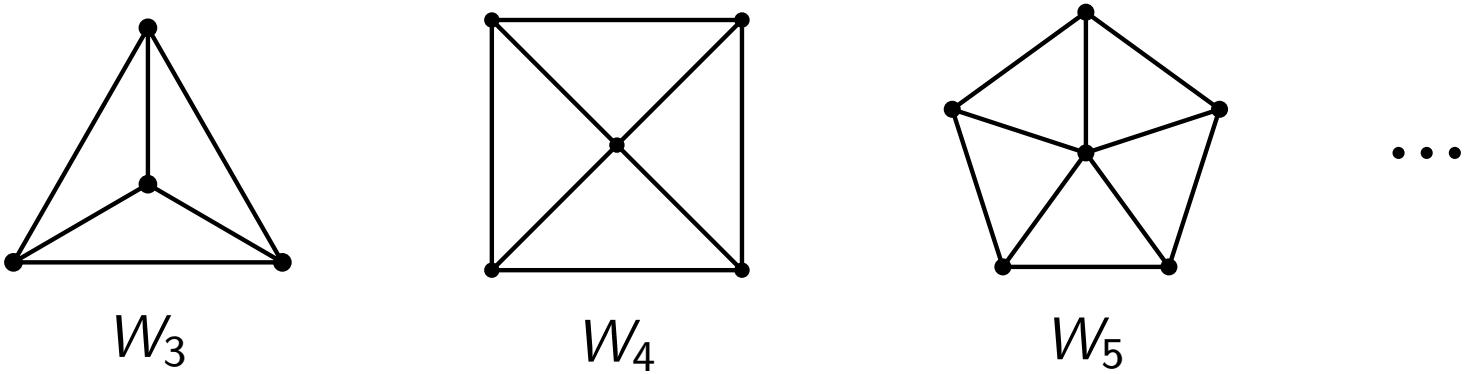
Linear combination with coefficients 2, 3, 6, 3, 4 spans:

$$H_{14}(\Delta_6) \cong \mathbb{Q}$$

Corollary

$$H^{15}(M_6) \neq 0.$$

More examples



- $d([W_g]) = 0$
- $[W_g] \neq 0$ for g odd

$$\boxed{\text{Is } [W_g] \text{ in } \mathrm{Im}(d)?}$$

- No for $g = 3, 5$
- No for $g = 7$ (computer calculation)

Theorem

$$[W_g] \notin \mathrm{Im}(\mathrm{d}) \textit{ for } g \textit{ odd.}$$

Idea of proof: The class $[W_g]$ is nonzero in $H_{2g-1}(\Delta_g)$ if it pairs nontrivially with some cocycle

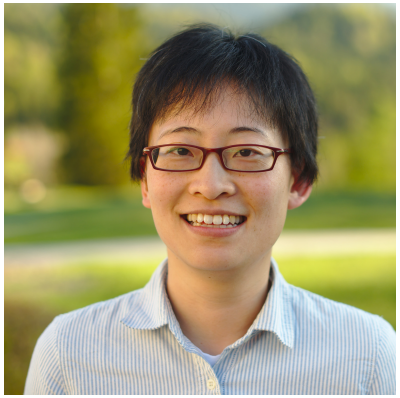
Following Drinfeld and Willwacher:

- $\alpha_G :=$ Feynman amplitude in $3d$ Chern-Simons theory
- Express as an integral over a configuration space
- Structured so that $\sigma_g = \sum_G \alpha_G [G]$ is a cocycle
- Compute, using tricks from analytic number theory:

$$\alpha_{W_g} \in \mathbb{Q}^\times \zeta(g)$$

Growth of $H^{4g-6}(M_g)$

Nonvanishing of $[W_g]$ implies $H^{4g-6}(M_g) \neq 0$ for g odd



Theorem (Chan, Galatius and P 2018)

$$\dim_{\mathbb{Q}} H^{4g-6}(M_g) > 1.3247^g + C.$$

Sketch of proof

- Subcomplex $\Delta_g^{lw} \subset \Delta_g$ parametrizing tropical curves with loop edges or vertices of positive weight is contractible
- $C_*(\Delta_g, \Delta_g^{lw})$ is isomorphic to the graph complex $K^{(g)}$
- $\prod_g H^*(\Delta_g)$ carries a Lie bracket, additive in g and $\#E(G)$

Theorem (Willwacher 2015)

$$\prod_g (H^{2g-1}(\Delta_g)) \cong \mathfrak{grt}_1.$$

Theorem (Brown 2012)

$$\mathfrak{grt}_1 \supset \mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots).$$

$$\dim_{\mathbb{Q}} \mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots)_g \gg \beta^g,$$

for $\beta < \beta_0 \approx 1.3247\ldots$ real root of $t^3 - t - 1$ $\square$

Unexpected

Conjecture (Kontsevich 1993, Church-Farb-Putman 2014)

$$H^{4g-4-k}(M_g) = 0, \quad \textit{for } g \gg k.$$

- Kontsevich stated six parallel conjectures, for the associative, commutative, and lie operads, with even and odd coefficients
- This is the "associative, odd" conjecture
- The "commutative, even" conjecture follows from vanishing of $H_k(\Delta_g)$ for $k < 2g - 1$

Further applications

Theorem (CGP 2018; Khoroshkin-Willwacher-Živković 2017)

The following cohomology groups do not vanish:

$$H^{21}(M_8), H^{23}(M_8), H^{27}(M_9), H^{27}(M_{10}), H^{31}(M_{10}).$$

Theorem (CGP 2018; Khoroshkin-Willwacher-Živković 2017)

The odd top weight cohomology grows exponentially:

$$\dim_{\mathbb{Q}} \bigoplus_{g'<g} \mathrm{Gr}^W_{6g'-6} H^{\mathrm{odd}}(M_{g'}) > 1.3247^g + C.$$

That's all folks

Vielen Dank!