COMPLEX ANALYTIC VANISHING CYCLES FOR FORMAL SCHEMES

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1. Previous work on vanishing cycles for formal schemes

Let k be a non-Archimedean field with nontrivial discrete valuation. A formal scheme over the ring of integers k° of k is said to be special if it is a locally finite union of open affine subschemes of the form $\mathrm{Spf}(A)$ with A isomorphic to a quotient of $k^\circ\{T_1, \ldots, T_m\}[[S_1, \ldots, S_n]]$. If all of these open affine subschemes can be found with $n = 0$, such $\mathfrak{X}$ is said to be of locally finite type (or of finite type if in addition $\mathfrak{X}$ is quasicompact). The class of locally finitely presented formal schemes $\mathfrak{X}$ is preserved under formal completion $\mathfrak{X}_{/\mathcal{Y}}$ of $\mathfrak{X}$ along an open subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, and the class of special formal schemes is preserved under formal completion of $\mathfrak{X}$ along an arbitrary subscheme of $\mathfrak{X}_s$. For example, if $\mathcal{X}$ is a scheme of finite type over k°, then the formal completion $\widehat{\mathcal{X}}$ (resp. $\widehat{\mathcal{X}}_{/\mathcal{Y}}$) of $\mathcal{X}$ along its closed fiber $\mathcal{X}_s = \mathcal{X} \otimes_{k^\circ} \widetilde{k}$ (resp. along an arbitrary subscheme $\mathcal{Y} \subset \mathcal{X}_s$) is a formal scheme of finite type (resp. a quasicompact special formal scheme) over k°. In what follows, we assume for simplicity that the residue field $\widetilde{k}$ is algebraically closed and all of the special formal schemes and schemes considered are quasicompact.

Each special formal scheme $\mathfrak{X}$ over k° has a generic fiber $\mathfrak{X}_\eta$, which is a paracompact strictly k-analytic space, and a closed fiber $\mathfrak{X}_s$, which is a scheme of finite type over the residue field $\widetilde{k}$ of k. In [Ber96] and [Ber15, §3.1], we constructed as follows a vanishing cycles functor $\Psi_\eta : \mathfrak{X}_\eta^\sim \to \mathfrak{X}_s(G)^\sim$ from the category of étale sheaves on $\mathfrak{X}_\eta$ to the category of étale sheaves on $\mathfrak{X}_s$ provided with a continuous (discrete) action of G, the Galois group of k. Recall that, by [Ber96, 2.1], the functor $\mathfrak{U} \mapsto \mathfrak{U}_s$ from the category special formal schemes étale over $\mathfrak{X}$ to that of schemes étale over $\mathfrak{X}_s$ is an equivalence of categories. We fix an inverse functor $\mathfrak{U}_s \mapsto \mathfrak{U}$ and, for a finite extension k' of k, set $\mathfrak{U}_{k'} = \mathfrak{U}\widehat{\otimes}_{k^\circ} k'^\circ$. Then for an étale sheaf F on $\mathfrak{X}_\eta$ and a scheme $\mathfrak{U}_s$ étale over $\widetilde{k}$, one has

$$\Psi_\eta(F)(\mathfrak{U}_s) = \varinjlim F((\mathfrak{U}_{k'})_\eta) \ ,$$

where the direct limit is taken over finite extensions k' of k in the algebraic closure k^{a} of k. In particular, for any discrete G-module Λ there is an associated complex of vanishing cycles sheaves $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ on $\mathfrak{X}_s$, where $\Lambda_{\mathfrak{X}_\eta}$ is the locally constant sheaf on $\mathfrak{X}_\eta$ induced by Λ. The construction is functorial and, therefore, any morphism of special formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to a morphism

$$\theta_\eta(\varphi, \Lambda) : \varphi_s^*(R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})) \to R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}) \ .$$

Among other things, we proved the following results. Suppose Λ is finite of order not divisible by $\mathrm{char}(\widetilde{k})$. Then

(i) the sheaves $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ are constructible;

(ii) one has $H^q(\mathfrak{X}_{\overline{\eta}}, \Lambda) = R^q\Gamma(\mathfrak{X}_s, R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta}))$, where $\mathfrak{X}_{\overline{\eta}} = \mathfrak{X}_\eta \widehat{\otimes}_k \widehat{k^{\mathrm{a}}}$;

(iii) given $\mathfrak{X}$ and $\mathfrak{Y}$ as above, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for any pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ congruent modulo $\mathcal{J}$, one has $\theta_\eta(\varphi, \Lambda) = \theta_\eta(\psi, \Lambda)$;

(iv) given a scheme $\mathcal{Y}$ of finite type over k° and a subscheme $\mathcal{Z} \subset \mathcal{Y}_s$, there is a canonical isomorphism $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta})\big|_{\mathcal{Z}} \widetilde{\to} R\Psi_\eta(\Lambda_{(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta})$, where $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta})$ is the vanishing cycles complex of the scheme $\mathcal{Y}$.

We remark that although the above functor Ψ_η gives rise to vanishing cycles complexes for arbitrary discrete G-modules Λ, e.g., $\mathbf{Z}$, those complexes do not possess good properties, and the reason is that such properties are not satisfied by the integral étale cohomology groups of algebraic varieties and non-Archimedean analytic spaces.

2. Complex analytic vanishing cycles for formal schemes

Let now K be a non-Archimedean field of the same type as k and, in addition, assume its ring of integers K° contains the field of complex numbers $\mathbf{C}$ and $\mathbf{C} \widetilde{\to} \widetilde{K}$. There is a canonical isomorphism $G \widetilde{\to} \varprojlim \mu_n$ of the Galois group G of K. The element $\sigma = (e^{\frac{2\pi i}{n}})_{n \geq 1}$ of the projective limit generates a subgroup Π isomorphic to $\mathbf{Z}$ and defines an isomorphism $G \widetilde{\to} \widehat{\mathbf{Z}}$. Furthermore, each generator ω of the maximal ideal $K^{\circ\circ}$ of K° induces a homomorphism $\mathcal{O}_{\mathbf{C},0} \to K^\circ$ that takes the coordinate function z of $\mathbf{C}$ to ω. It gives rise to isomorphisms $\widehat{\mathcal{O}}_{\mathbf{C},0} \widetilde{\to} K^\circ$ and $G = \mathrm{Gal}(K^{\mathrm{a}}/K) \widetilde{\to} \mathrm{Gal}(\mathcal{K}^{\mathrm{a}}/\mathcal{K})$, where $\mathcal{K}$ is the fraction field of $\mathcal{O}_{\mathbf{C},0}$. The latter isomorphism does not depend on the choice of ω and identifies the subgroup Π with the fundamental group $\pi_1(\mathbf{C}^*)$.

In our work in progress we construct, for every special formal scheme $\mathfrak{X}$ over K°, an exact functor

$$D^b(\Pi\text{-Mod}) \to D^b(\mathfrak{X}_s^h(\Pi)) : \Lambda^\cdot \mapsto R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) ,$$

where the former denotes the derived category of bounded complexes of Π-modules, and the latter denotes the derived category of bounded complexes of abelian Π-sheaves on $\mathfrak{X}_s^h$, the complex analytification of the scheme $\mathfrak{X}_s$. (The notation $R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{X}_\eta})$ for the resulting complex is suggestive, even $\Lambda^\cdot_{\mathfrak{X}_\eta}$ does not represent a complex of étale sheaves on $\mathfrak{X}_\eta$ unless $\Lambda^\cdot$ is a complex of discrete G-modules.) We prove that the complexes $R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{X}_\eta})$ possess the following properties:

(i) they are functorial in $\mathfrak{X}$ and, in particular, every morphism of special formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to a morphism of complexes

$$\theta_\eta^h(\varphi, \Lambda^\cdot) : \varphi_s^{h*}(R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{X}_\eta})) \to R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{Y}_\eta}) ;$$

(ii) there is a canonical isomorphism

$$R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) = R\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta}) \otimes^{\mathbf{L}}_{\mathbf{Z}_{\mathfrak{X}_s^h}} \underline{\Lambda}^\cdot_{\mathfrak{X}_s^h} ,$$

where $\underline{\Lambda}^\cdot_{\mathfrak{X}_s^h}$ is the complex of constant sheaves on $\mathfrak{X}_s^h$ associated to the complex $\Lambda^\cdot$ and provided with the induced action of the group Π;

(iii) the sheaves $R^q\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ are (algebraically) constructible in the sense of [Ver76, §2], and the action of Π on them is quasi-unipotent;

(iv) given a subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, there is a canonical isomorphism

$$R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})\big|_{\mathcal{Y}} \widetilde{\rightarrow} R\Psi^h_\eta(\Lambda^\cdot_{(\mathfrak{X}_{/\mathcal{Y}})_\eta}) \ ;$$

(v) given $\mathfrak{X}$ with rig-smooth generic fiber, there exists $n \geq 1$ such that, for every $\mathfrak{Y}$ of finite type over K°, every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ congruent modulo $(K^{\circ\circ})^n$ and every $\Lambda^\cdot$, one has $\theta^h_\eta(\varphi, \Lambda^\cdot) = \theta^h_\eta(\psi, \Lambda^\cdot)$;

(vi) given $\mathfrak{X}$ and $\mathfrak{Y}$ both with rig-smooth generic fibers, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ congruent modulo $\mathcal{J}$ and every $\Lambda^\cdot$, one has $\theta^h_\eta(\varphi, \Lambda^\cdot) = \theta^h_\eta(\psi, \Lambda^\cdot)$;

(vii) if $\Lambda^\cdot$ is a complex of discrete $\mathbf{Z}/n\mathbf{Z}[G]$-modules whose cohomology modules are finite, then there is a canonical isomorphism

$$(R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}))^h \widetilde{\rightarrow} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ ,$$

where $R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$ is the vanishing cycles complex on $\mathfrak{X}_s$ from §1;

(viii) given a morphism of germs of complex analytic spaces $(B, b) \to (\mathbf{C}, 0)$, a scheme $\mathcal{Y}$ of finite type over $\mathcal{O}_{B,b}$, and a generator ω of $K^{\circ\circ}$, there is a canonical isomorphism

$$R\Psi_\eta(\Lambda^\cdot_{\mathcal{Y}^h_\eta}) \widetilde{\rightarrow} R\Psi^h_\eta(\Lambda^\cdot_{\widehat{\mathcal{Y}}_\eta}) \ .$$

Here is an explanation of the objects on both sides of the isomorphism in (viii).

First of all, the element ω defines an isomorphism $\widehat{\mathcal{O}}_{\mathbf{C},0} \widetilde{\rightarrow} K^\circ$ which allows one to view the formal completion $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$ along the closed fiber $\mathcal{Y}_s = \mathcal{Y} \otimes_{\mathcal{O}_{B,b}} \mathbf{C}$ as a special formal scheme over K°, and the right hand side in (viii) is the value at $\Lambda^\cdot$ of the above exact functor $R\Psi^h_\eta$ associated to the special formal scheme $\widehat{\mathcal{Y}}$.

Furthermore, the scheme $\mathcal{Y}$ defines a complex analytic space $\mathcal{Y}^h$ over an open neighborhood of b in B. If the neighborhood is small enough, there is an induced morphism $\mathcal{Y}^h \to \mathbf{C}$. The same construction applied to the schemes $\mathcal{Y}_s$ and $\mathcal{Y}_\eta = \mathcal{Y} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathcal{K}$ gives the usual complex analytification $\mathcal{Y}^h_s$ of $\mathcal{Y}_s$ and a space $\mathcal{Y}^h_\eta$, which can be identified with the preimage of $\mathbf{C}^*$ under the above morphism. The complex of Π-modules $\Lambda^\cdot$ defines a complex of locally constant sheaves on $\mathbf{C}^*$ whose pullback on $\mathcal{Y}^h_\eta$ is denoted by $\Lambda^\cdot_{\mathcal{Y}^h_\eta}$. The left hand side in (viii) is the value at $\Lambda^\cdot_{\mathcal{Y}^h_\eta}$ of the derived functor of the following complex analytic vanishing cycles functor Ψ_η from the category of sheaves on $\mathcal{Y}^h_\eta$ to the category of Π-sheaves on $\mathcal{Y}^h_s$ (it is a particular case of the definition from [SGA7, Exp. XIV]). The above three analytic spaces define morphisms

$$\begin{array}{ccccc} \mathcal{Y}^h_\eta & \xrightarrow{j} & \mathcal{Y}^h & \xleftarrow{i} & \mathcal{Y}^h_s \\ \uparrow & \nearrow_{\overline{j}} & & & \\ \mathcal{Y}^h_{\overline{\eta}} & & & & \end{array}$$

where $\mathcal{Y}^h_{\overline{\eta}} = \mathcal{Y}^h_\eta \times_{\mathbf{C}^*} \mathbf{C}$ and the fiber product is taken with respect to the universal covering map $\mathbf{C} \to \mathbf{C}^* : z \mapsto e^{2\pi i z}$. The complex analytic vanishing cycles functor is defined by $\Psi_\eta(F) = i^*(\overline{j}_* \overline{F})$, where $\overline{F}$ is the lift of F to $\mathcal{Y}_{\overline{\eta}}$.

The continuity properties (v) and (vi) are stronger than corresponding results from [Ber96] and [Ber15], but the assumptions on rig-smoothness are probably superfluous. In any case, if $\mathfrak{X} = \widehat{\mathcal{Y}}$ for $\mathcal{Y}$ from (viii), then $\mathfrak{X}_\eta$ is rig-smooth if and if the complex analytic space $\mathcal{Y}^h_\eta$ is smooth over $\mathbf{C}^*$.

The main ingredients used in the construction of the vanishing cycles complexes and establishing their properties are Michael Temkin's work on desingularization of quasi-excellent schemes in characteristic zero ([Tem08], [Tem09]), the work of Kazuya Kato and his collaborators on log geometry ([Kat89], [KN99], [Nak98]), and author's work on vanishing cycles for formal schemes ([Ber93], [Ber96b], [Ber15]).

3. Integral étale cohomology of analytic spaces

The above results are used to define, for every compact strictly K-analytic space X, *integral étale cohomology groups* $H^q(\overline{X}, \mathbf{Z})$ of $\overline{X} = X\widehat{\otimes}_K\widehat{K^{\mathrm{a}}}$. Namely, we fix for every X a formal scheme $\mathfrak{X}$ of finite type over K° with $\mathfrak{X}_\eta = X$ (it exists by Raynaud's theory), and set $H^q(\overline{X}, \mathbf{Z}) = R^q\Gamma(\mathfrak{X}^h_s, R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}))$. (This definition corresponds to the property (ii) from §1.) We prove that

(i) the groups $H^q(\overline{X}, \mathbf{Z})$ do not depend on the choice of $\mathfrak{X}$ up to a canonical isomorphism, and the correspondence $X \mapsto H^q(\overline{X}, \mathbf{Z})$ is functorial in X;

(ii) the groups $H^q(\overline{X}, \mathbf{Z})$ are finitely generated and provided with a quasi-unipotent action of Π;

(iii) given a finite covering $\mathcal{V} = \{V_i\}_{i\in I}$ of X by compact strictly analytic domains, there is a Leray spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{V}, \mathcal{H}^q) \Longrightarrow H^{p+q}(\overline{X}, \mathbf{Z}) \ ,$$

where $\mathcal{H}^q$ is the presheaf $V \mapsto H^q(\overline{V}, \mathbf{Z})$ on the family of those domains;

(iv) for every prime l, there are canonical Π-equivariant isomorphisms

$$H^q(\overline{X}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_l \widetilde{\to} H^q(\overline{X}, \mathbf{Z}_l) = \varprojlim H^q(\overline{X}, \mathbf{Z}/l^n\mathbf{Z}) \ ,$$

where $H^q(\overline{X}, \mathbf{Z}/l^n\mathbf{Z})$ are the étale cohomology groups of $\overline{X}$ from [Ber93];

(v) if X is rig-smooth, the uniform space of morphisms of compact strictly K-analytic spaces $Y \to X$ acts continuously on the discrete set of induced homomorphisms $H^q(\overline{X}, \mathbf{Z}) \to H^q(\overline{Y}, \mathbf{Z})$;

(vi) there are canonical Π-equivariant homomorphisms $H^q(|\overline{X}|, \mathbf{Z}) \to H^q(\overline{X}, \mathbf{Z})$, where the groups on the left hand side are cohomology groups of the underlying topological space $|\overline{X}|$ of $\overline{X}$;

(vii) in the situation of (viii) from §2, if $\mathcal{Y}$ is separated and $\mathcal{Y} = \mathcal{Y}_\eta$, then every morphism of K-analytic spaces $X \to \mathcal{Y}^{\mathrm{an}}$ gives rise to canonical Π-equivariant homomorphisms $H^q(\overline{\mathcal{Y}^h}, \mathbf{Z}) \to H^q(\overline{X}, \mathbf{Z})$, which are functorial in X and $\mathcal{Y}$.

In (iv), if $X = \mathcal{Y}^{\mathrm{an}}$ for a proper scheme $\mathcal{Y}$ over K then, by [Ber93], $H^q(\overline{X}, \mathbf{Z}_l)$ coincide with the l-adic étale cohomology groups $H^q(\overline{\mathcal{Y}}, \mathbf{Z}_l)$ of the scheme $\overline{\mathcal{Y}} = \mathcal{Y} \otimes_K K^{\mathrm{a}}$, and we get canonical Π-equivariant isomorphisms

$$H^q(\overline{\mathcal{Y}^{\mathrm{an}}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_l \widetilde{\to} H^q(\overline{\mathcal{Y}}, \mathbf{Z}_l) \ .$$

In (v), the uniform space structure on the set of morphisms is from [Ber94, §6].

In (vii), $\mathcal{Y}^{\mathrm{an}}$ is the K-analytic space associated (in [Ber15, §3.2]) to the scheme $\mathcal{Y} \otimes_{\mathcal{O}_{B,b}} (\widehat{\mathcal{O}}_{B,b} \otimes_{K^\circ} K)$, and $\overline{\mathcal{Y}^h} = \mathcal{Y}^h \times_{\mathbf{C}^*} \mathbf{C}$ with respect to the morphism $\mathbf{C} \to \mathbf{C}^* : z \mapsto e^{2\pi i z}$. If $\mathcal{Y}$ is proper over $\mathcal{K}$, then $\mathcal{Y}^{\mathrm{an}}$ is compact, and (vi) implies that there are canonical Π-equivariant isomorphisms

$$H^q(\overline{\mathcal{Y}^h}, \mathbf{Z}) \widetilde{\to} H^q(\overline{\mathcal{Y}^{\mathrm{an}}}, \mathbf{Z}) \ .$$

We conjecture that the above groups $H^q(\overline{X}, \mathbf{Z})$ are provided with a mixed Hodge structure which is functorial in X and such that, if $X = \mathcal{Y}^{\rm an}$ for a proper scheme $\mathcal{Y}$ over $\mathcal{K}$ as in the previous paragraph, it coincides with the limit mixed Hodge structure on the groups $H^q(\overline{\mathcal{Y}^h}, \mathbf{Z})$.

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