THE NON-ARCHIMEDEAN MONGE-AMPÈRE EQUATION

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ABSTRACT. These are lecture notes for the talks by the authors at the 2015 Simons Symposium on Tropical and Nonarchimedean Geometry. We outline the joint work of the authors with Charles Favre on the solution to the non-Archimedean Monge-Ampère equation, comparing it to its complex counterpart.

INTRODUCTION

The purpose of these notes is to discuss the Monge-Ampère equation

$$\mathrm{MA}(\phi) = \mu$$

in both the complex and non-Archimedean setting. Here μ is a positive measure[1] on the analytification of a smooth projective variety, ϕ is a semipositive metric on an ample line bundle on X, and MA is the Monge-Ampère operator. All these terms will be explained below.

In the non-Archimedean case, our presentation is based on the papers [BFJ12, BFJ14] joint with Charles Favre, to which we refer for details. In the complex case, we follow [BBGZ13] rather closely.

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1. METRICS ON LINES BUNDLES

Let K be a field equipped with a complete multiplicative norm and let X be a smooth projective variety over K. To this data we can associate an analytification X^{an}. When K is the field of complex numbers with its usual norm, X^{an} is a compact complex manifold. When the norm is non-Archimedean, X^{an} is a K-analytic space in the sense of Berkovich [Ber90]. In either case, it is a compact Hausdorff space.

Let L be a line bundle on X. It also admits an analytification L^{an}. A *metric* on L^{an} is a rule that to a local section $s : U \to L^{\mathrm{an}}$, where $U \subset X^{\mathrm{an}}$, associates a function $\|s\|$ on U, subject to the condition $\|fs\| = |f| \cdot \|s\|$, for any analytic function f on U. The metric is *continuous* if $\|s\|$ is continuous for every s.

For our purposes it is convenient to use additive notation for metrics and line bundles. Given an open cover U_α of X^{an} and local trivializations of L^{an} on each

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[1]All measures in this paper will be assumed to be Radon measures.

U_α, we can identify a section s of L with a collection $(s_\alpha)_\alpha$ of analytic functions. A metric ϕ is then a collection of functions $(\phi_\alpha)_\alpha$ in such a way that $\|s\|_\phi = |s_\alpha| e^{-\phi_\alpha}$ on U_α. With this convention, if ϕ is a metric on L^{an}, any other metric is of the form $\phi + f$, where f is a function on X^{an}. If ϕ_i is a metric on L_i, $i = 1, 2$, then $\phi_1 + \phi_2$ is a metric on $L_1 + L_2$.

Over the complex numbers, smooth metrics ϕ (i.e. each ϕ_α is smooth), play an important role. Of similar status, for K is non-Archimedean, are *model metrics* defined as follows.[2] Let R be the valuation ring of K and k the residue field. A *model* of X is a normal scheme $\mathcal{X}$, flat and projective over $S := \operatorname{Spec} R$ and with generic fiber isomorphic to X. A model of L is a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ whose restriction to X is isomorphic to L. It defines a continuous metric $\phi_{\mathcal{L}}$ on L in such a way that a local nonvanishing section of $\mathcal{L}$ has norm constantly equal to one. Model functions, that is, model metrics on $\mathcal{O}_X$, are dense in $C^0(X^{\mathrm{an}})$. We refer to [CL11] or [BFJ12] for a more thorough discussion.

Over $\mathbf{C}$, a smooth metric ϕ on L^{an} is semipositive (positive) if its curvature form $dd^c\phi$ is a semipositive (positive) $(1,1)$-form. Here $dd^c\phi = dd^c\phi_\alpha = \frac{i}{\pi}\partial\overline{\partial}\phi_\alpha$ for any α. Such metrics only exist when L is nef.

In the non-Archimedean setting we say that a model metric $\phi_{\mathcal{L}}$ on L^{an} is semipositive if the line bundle $\mathcal{L}$ is relatively nef, that is, its degree is nonnegative on any proper curve contained in the special fiber $\mathcal{X}_0$. This implies that L is nef.

In both the complex and non-Archimedean case we say that a continuous metric ϕ is semipositive if there exists a sequence $(\phi_m)_1^\infty$ of semipositive smooth/model metrics such that $\lim_{m\to\infty} \sup_{X^{\mathrm{an}}} |\phi_m - \phi| = 0$. In the non-Archimedean case, this notion was first introduced by Zhang [Zha95] and Gubler [Gub98]. In the complex case, it is more natural to say that a continuous metric ϕ is semipositive if its curvature current $dd^c\phi$ is a positive closed current. At least when L is ample, one can then prove (see §6 below) that ϕ can be approximated by smooth metrics; such an approximation is furthermore crucial for many arguments in pluripotential theory.

In the non-Archimedean case, Chambert-Loir and Ducros have introduced a notion of forms and currents on Berkovich spaces. However, it is not known whether a continuous metric whose curvature current (in their sense) is semipositive can be approximated by semipositive model metrics.

In both the complex and non-Archimedean case we denote by $\mathrm{PSH}^0(L^{\mathrm{an}})$ the space of continuous semipositive metrics on L^{an}. Here the superscript refers to continuity (C^0) whereas "PSH" reflects the fact that in the complex case, semipositive metrics are global versions of plurisubharmonic functions.

2. The Monge-Ampère operator

In the complex case, the Monge-Ampère operator is a second order differential operator: we set $\mathrm{MA}(\phi) = (dd^c\phi)^n$ for a smooth metric ϕ. It is a nonlinear operator if $n > 1$. When ϕ is semipositive, $\mathrm{MA}(\phi)$ is a smooth positive measure on X^{an} of mass (L^n). It is a *volume form*, that is, equivalent to Lebesgue measure, if ϕ is positive.

[2]Model metrics are not smooth in the sense of [CD12] but nevertheless, for our purposes, play the same role as smooth metrics in the complex case.

Next we turn to the non-Archimedean setting. *From now on we assume that K is discretely valued.* Pick a uniformizer t of the maximal ideal in the valuation ring R of K. Thus $R \simeq k[\![t]\!]$ and $K \simeq k(\!(t)\!)$, where k is the residue field of K.

Consider a model metric $\phi_{\mathcal{L}}$, associated to a model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $S = \operatorname{Spec} R$. Write the special fiber as $\mathcal{X}_0 = \operatorname{div} t = \sum_{i \in I} b_i E_i$, where E_i are the irreducible components of $\mathcal{X}_0$ and $b_i \in \mathbf{Z}_{>0}$. To each E_i is associated a unique (divisorial) point $x_i \in X^{\mathrm{an}}$. We then define

$$\mathrm{MA}(\phi) := \sum_{i \in I} b_i (\mathcal{L}|_{E_i})^n \delta_{x_i}.$$

If $\phi_{\mathcal{L}}$ is semipositive, $\mathcal{L}|_{E_i}$ is nef; hence $(\mathcal{L}|_{E_i})^n \geq 0$ and $\mathrm{MA}(\phi_{\mathcal{L}})$ is a positive measure. Its total mass is

$$\int_{X^{\mathrm{an}}} 1 \cdot \mathrm{MA}(\phi_{\mathcal{L}}) = \sum_{i \in I} b_i (\mathcal{L}|_{E_i})^n = (\mathcal{L}^n \cdot \mathcal{X}_0) = (\mathcal{L}^n \cdot \mathcal{X}_\eta) = (L^n).$$

Here the second to last equality follows from the flatness of $\mathcal{X}$ over S, and the last equality from $\mathcal{X}_\eta \simeq X$, $\mathcal{L}_\eta \simeq L$.

From now on assume that L is *ample*, that is, we have a *polarized* pair (X, L). In both the complex and non-Archimedean case we define $\mathrm{MA}(\phi)$ for a continuous semipositive metric by $\mathrm{MA}(\phi) := \lim_{m \to \infty} \mathrm{MA}(\phi_m)$ for any sequence $(\phi_m)_1^\infty$ converging uniformly to ϕ. Of course, it is not obvious that the limit exists or independent of the sequence $(\phi_m)_1^\infty$. In the complex case this is a very special case of the Bedford-Taylor theory developed in [BT82, BT87]. The analogous construction in the non-Archimedean case is due to Chambert-Loir [CL06].

3. The complex Monge-Ampère equation

Theorem 3.1. *Let (X, L) be a polarized complex projective variety of dimension n and let μ be a positive measure on X^{an} of total mass (L^n).*

- (i) *If μ is a volume form, then there exists a smooth positive metric ϕ on L^{an} such that $\mathrm{MA}(\phi) = \mu$.*
- (ii) *If μ is absolutely continuous with respect to Lebesgue measure, with density in L^p for some $p > 1$, then there exists a (Hölder) continuous metric ϕ on L^{an} such that $\mathrm{MA}(\phi) = \mu$.*
- (iii) *The metrics in (i) and (ii) are unique up to additive constants.*

The uniqueness statement in the setting of (i) is due to Calabi. The much harder existence part was proved by Yau [Yau78], using PDE techniques. The combined result is often called the Calabi-Yau Theorem.

The general setting of (ii)–(iii) was treated by Kołodziej [Koł98, Koł03] who used methods of pluripotential theory together with a nontrivial reduction to Yau's result. Guedj and Zeriahi [GZ07] more generally established the existence of solutions of $\mathrm{MA}(\phi) = \mu$ for positive measures μ (of mass (L^n)) that do not put mass on pluripolar sets. In this generality, the metrics ϕ are no longer continuous but rather lie in a suitable energy class, modeled upon work by Cegrell [Ceg98]. Dinew [Din09], improving upon an earlier result by Błocki [Bło03], proved the corresponding uniqueness theorem. All these existence and uniqueness results are furthermore valid (in a suitable formulation) in the transcendental case, when (X, ω) is a Kähler manifold.

The complex Monge-Ampère equation is of fundamental importance to complex geometry. For example, it implies that every compact complex manifold with vanishing first Chern class (such manifolds are now called Calabi-Yau manifolds) admit a Ricci flat metric in any given Kähler class. The complex Monge-Ampère equation also plays a key role in recent work on the space of Kähler metrics.

4. The non-Archimedean Monge-Ampère equation

As before, suppose K is a discretely valued field with valuation ring R and residue field k. We further assume that K has *residue characteristic zero*, $\operatorname{char} k = 0$. In this case, X admits SNC models, that is, regular models $\mathcal{X}$ such that the special finer $\mathcal{X}_0$ has simple normal crossings. The dual complex $\Delta_{\mathcal{X}}$, encoding intersections between irreducible components of $\mathcal{X}_0$, then embeds as a compact subset of X^{an}.

Theorem 4.1. *Let (X, L) be a polarized complex projective variety of dimension n over K. Assume X is defined over a smooth k-curve. Let μ be a positive measure on X^{an} of total mass (L^n), supported on the dual complex of some SNC model.*

(i) *There exists a continuous metric ϕ on L^{an} such that* $\mathrm{MA}(\phi) = \mu$.
(ii) *The metric in (i) is unique up to an additive constant.*

Here the condition on X means that there exists a smooth projective curve C over k, a smooth projective variety Y over C, and a point $p \in C$ such that X is isomorphic to the base change $Y \times_k \operatorname{Spec} K$, where K is the fraction field of $\widehat{\mathcal{O}}_{C,p}$. This condition is presumably redundant, but is used in the proof: see §8.

To our knowledge, the first to consider the Monge-Ampère equation (or Calabi-Yau problem) in a non-Archimedean setting were Kontsevich and Tschinkel [KT00]. They outlined a strategy in the case when μ is a point mass.

The case of curves ($n = 1$) was treated in detail by Thuillier in his thesis [Thu05]; see also [BR10, FJ04]. In this case, the Monge-Ampère equation is linear and one can construct fundamental solutions by exploring the topological structure of X^{an}.

In higher dimensions, Yuan and Zhang [YZ13] proved the uniqueness statement (ii). Their proof, based on the method by Błocki, is valid in a more general context than stated above. The first existence result was obtained by Liu [Liu11], who treated the case when X is a maximally degenerate abelian variety and μ is equivalent to Lebesgue measure on the skeleton of X. Curiously, his proof relies on Yau's result. The existence result (i) above was proved by Favre and the authors in [BFJ14] and the companion paper [BFJ12]. We will discuss our approach below.

The geometric ramifications of the non-Archimedean Monge-Ampère equations remain to be developed.

5. A variational approach

We shall present a unified approach to solving the complex and non-Archimedean Monge-Ampère equations in any dimension. The method goes back to Alexandrov's work in convex geometry [Ale38]. It was adapted to the complex case in [BBGZ13] and to the non-Archimedean analogue in [BFJ14].

The general strategy is to construct an *energy functional*

$$E : \mathrm{PSH}^0(L^{\mathrm{an}}) \to \mathbf{R}$$

whose derivative is the Monge-Ampère operator, $E' = \mathrm{MA}$, in the sense that

$$\frac{d}{dt}E(\phi+tf)|_{t=0} = \int_{X^{\mathrm{an}}} f\,\mathrm{MA}(\phi),$$

for every continuous semipositive metric $\phi \in \mathrm{PSH}^0(L^{\mathrm{an}})$ and every smooth/model function f on X^{an}.

Grant the existence of this functional for the moment. Given a measure μ on X^{an}, consider the functional $F_\mu : \mathrm{PSH}^0(L^{\mathrm{an}}) \to \mathbf{R}$ defined by

$$F_\mu(\phi) = E(\phi) - \int \phi\,\mu.$$

Suppose we can find $\phi \in \mathrm{PSH}^0(L^{\mathrm{an}})$ that maximizes F_μ. Since the derivative of F_μ is equal to $F'_\mu = \mathrm{MA} - \mu$, we then have $0 = F'_\mu(\phi) = \mathrm{MA}(\phi) - \mu$ as required.

Now, there are at least three problems with this approach:

(1) There is a priori no reason why a maximizer should exist in $\mathrm{PSH}^0(L^{\mathrm{an}})$. We resolve this by introducing a larger space $\mathrm{PSH}(L^{\mathrm{an}})$ with suitable compactness properties and find a maximizer there.
(2) Granted the existence of a maximizer $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$, we are maximizing over a convex set rather than a vector space, so there is no reason why $F'_\mu(\phi) = 0$. Compare maximizing the function $f(x) = x^2$ on the interval $[-1, 1]$: the maximum is not at a critical point.
(3) In the end we want to show that—after all—the maximizer is continuous, that is, $\phi \in C^0(L^{\mathrm{an}})$.

We shall discuss how to address (1) and (2) in the next two sections. The continuity result in (3) requires a priori capacity estimates due to Kołodziej, and will not be discussed in these notes.

6. Singular semipositive metrics

Plurisubharmonic (psh) functions are among the *objets souples* (soft objects) in complex analysis according to P. Lelong [Lel85]. This is reflected in certain useful compactness properties. The global analogues of psh functions are semipositive singular metrics on holomorphic line bundles. Here “singular” means that vectors may have infinite length.

Theorem 6.1. *Let K be either $\mathbf{C}$ or a discretely valued field of residue characteristic zero, and let (X, L) be a smooth projective polarized variety over K. Then there exists a unique class $\mathrm{PSH}(L^{\mathrm{an}})$, the set of singular semipositive metrics, with the following properties:*

- $\mathrm{PSH}(L^{\mathrm{an}})$ *is a convex set which is closed under maxima and addition of constants;*
- $\mathrm{PSH}(L^{\mathrm{an}}) \cap C^0(L^{\mathrm{an}}) = \mathrm{PSH}^0(L^{\mathrm{an}})$*;*
- *if s_i, $1 \le i \le p$, are nonzero global sections of mL for some $m \ge 1$, then $\phi := \frac{1}{m}\max_i \log|s_i| \in \mathrm{PSH}(L^{\mathrm{an}})$; further, ϕ is continuous iff the sections s_i have no common zero.*
- *if (ϕ_j) is an arbitrary family in $\mathrm{PSH}(L^{\mathrm{an}})$ that is uniformly bounded from above, then the usc regularization of $\sup_j \phi_j$ belongs to $\mathrm{PSH}(L^{\mathrm{an}})$;*
- *if (ϕ_j) is a decreasing net in $\mathrm{PSH}(L^{\mathrm{an}})$, then either $\phi_j \to -\infty$ uniformly on X^{an}, or $\phi_j \to \phi$ pointwise on X^{an} for some $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$;*

- ***Regularization**: for every $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ there exists a decreasing sequence $(\phi_m)_{m=1}^{\infty}$ of smooth/model metrics such that ϕ_m converges pointwise to ϕ on X^{an} as $m \to \infty$;*
- ***Compactness**: the space $\mathrm{PSH}(L^{\mathrm{an}})/\mathbf{R}$ is compact.*

To make sense of the compactness statement we need to specify the topology on $\mathrm{PSH}(L^{\mathrm{an}})$. In the complex case, one usually fixes a volume form μ on X^{an} and takes the topology induced by the L^1-norm: $\|\phi - \psi\| = \int_{X^{\mathrm{an}}} |\phi - \psi|\mu$. In the non-Archimedean case, there is typically no volume form on X^{an}. Instead, we say that a net $(\phi_j)_j$ in $\mathrm{PSH}(L^{\mathrm{an}})$ converges to ϕ if $\lim_j \sup_{\Delta_{\mathcal{X}}} |\phi_j - \phi| = 0$ for every SNC model $\mathcal{X}$. Implicit in this definition is that the restriction to $\Delta_{\mathcal{X}}$ of every singular metric in $\mathrm{PSH}(L^{\mathrm{an}})$ is continuous: see Theorem 6.2 below.

In the complex case, one typically defines $\mathrm{PSH}(L^{\mathrm{an}})$ as the set of usc singular metrics ϕ that are locally represented by L^1 functions and whose curvature current $dd^c\phi$ (computed in the sense of distributions) is a positive closed current. Thus ϕ is locally given as the sum of a smooth function and a psh function. Most of the statements above then follow from basic facts about plurisubharmonic functions in $\mathbf{C}^n$. The regularization result is the most difficult. On $\mathbf{C}^n$ it is easy to regularize using convolutions. With some care, one can in the global (projective) case glue together local regularizations to obtain a global one. See [Dem92] for a general result and [BK07] for a relatively simple argument applicable in our setting.

In the non-Archimedean case, we are not aware of any workable a priori definition of $\mathrm{PSH}(L^{\mathrm{an}})$. Chambert-Loir and Ducros have a notion of forms and currents on Berkovich spaces, but it is unclear if it gives the right objects for the purposes of the theorem above. Instead, we prove the following result:

Theorem 6.2. *For any SNC model $\mathcal{X}$, the restriction of the dual complex $\Delta_{\mathcal{X}} \subset X^{\mathrm{an}}$ of the set of model metrics on L^{an} forms an equicontinuous family.*

This is proved using a rather subtle argument, involving intersection numbers on toroidal models dominating $\mathcal{X}$. It would be interesting to have a different argument. At any rate, Theorem 6.2 allows us to define $\mathrm{PSH}(L^{\mathrm{an}})$ as the set of usc singular metrics ϕ satisfying, for every sufficiently large SNC model $\mathcal{X}$,

(i) $(\phi - \phi_0) \circ r_{\mathcal{X}} \geq \phi - \phi_0$;
(ii) the restriction of ϕ to $\Delta_{\mathcal{X}}$ is a uniform limits of a sequence $\phi_m|_{\Delta_{\mathcal{X}}}$, where each ϕ_m is a semipositive model metric.

Here ϕ_0 is a fixed model metric, determined by some model dominated by $\mathcal{X}$. The map $r_{\mathcal{X}} : X^{\mathrm{an}} \to \Delta_{\mathcal{X}} \subset X^{\mathrm{an}}$ is a natural retraction. Since ϕ is usc, condition (i) implies that $\phi = \phi_0 + \lim_{\mathcal{X}}(\phi - \phi_0) \circ r_{\mathcal{X}}$, so that ϕ is determined by its restrictions to all dual complexes.

With this definition, the compactness of $\mathrm{PSH}(L^{\mathrm{an}})/\mathbf{R}$ follows from Theorem 6.2 and Ascoli's theorem. Regularization, however, is quite difficult to show. We are not aware of any procedure that would replace convolution in the complex case. Instead we use algebraic geometry. Here is an outline of the proof.

Fix $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$. For any SNC model $\mathcal{X}$, ϕ naturally induces a model metric $\phi_{\mathcal{X}}$. The semipositivity of ϕ implies that the net $(\phi_{\mathcal{X}})_{\mathcal{X}}$, indexed by the collection of (isomorphism classes of) SNC models decreases to ϕ. Unfortunately, except in the curve case $n = 1$, $\phi_{\mathcal{X}}$ has no reason to be semipositive; this reflects the fact that the pushforward of a nef line bundle may fail to be nef. We address this by defining $\psi_{\mathcal{X}}$ as the supremum of all semipositive (singular) metrics dominated by

$\phi_{\mathcal{X}}$. We then show that $\psi_{\mathcal{X}}$ is continuous and can be *uniformly* approximated by a sequence $(\psi_{\mathcal{X},m})_m^\infty$ of semipositive model metrics. From this data it is not hard to produce a decreasing net of semipositive model metrics converging to ϕ.

Let us say a few words on the construction of the semipositive model metrics $\phi_{\mathcal{X},m}$ since this is a key step in the paper [BFJ12]. For simplicity assume that L is base point free and that $\psi_{\mathcal{X}}$ is associated to a line bundle $\mathcal{L}$ (rather than an **R**-line bundle) on $\mathcal{X}$. Let $\mathfrak{a}_m$ be the base ideal of $\mathcal{L}^{\otimes m}$, cut out by the global sections; it is cosupported on the special fiber $\mathcal{X}_0$. The sequence $(\mathfrak{a}_m)_m$ is a *graded sequence* in the sense that $\mathfrak{a}_l \cdot \mathfrak{a}_m \subset \mathfrak{a}_{m+n}$, Each $\mathfrak{a}_m$ naturally defines a semipositive model metric $\psi_{\mathcal{X},m}$ on L^{an}. The fact that $\psi_{\mathcal{X},m}$ converges *uniformly* to $\psi_{\mathcal{X}}$ translates into a statement that the graded sequence $(\mathfrak{a}_m)_m$ is "almost" finitely generated. This in turns is proved using *multiplier ideals* and ultimately reduces to the Kodaira vanishing theorem; to apply the latter it is crucial to work in residue characteristic zero.

The argument above proves that any $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ is the limit of a decreasing *net* of semipositive model metrics. When ϕ is continuous, the convergence is uniform by Dini's Theorem, and we can use the sup-norm to extract a decreasing *sequence* of model metrics converging to ϕ. In the general case, the *Monge-Ampère capacity* developed in [BFJ14, §4] (and modeled on [BT82, GZ05]) can similarly be used to extract a sequence.

7. Energy

In the complex case, the (Aubin-Mabuchi) energy functional is defined as follows. Fix a smooth semipositive reference metric ϕ_0 and set

$$E(\phi) := \frac{1}{n+1}\sum_{j=0}^{n}\int_{X^{\mathrm{an}}}(\phi-\phi_0)(dd^c\phi)^j\wedge(dd^c\phi_0)^{n-j}. \tag{7.1}$$

for any smooth metric ϕ. Here $(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}$ is a *mixed Monge-Ampère measure*. It is a positive measure if ϕ is semipositive.

In the non-Archimedean case, mixed Monge-Ampère measures can be defined using intersection theory when ϕ and ϕ_0 are model metrics, and the energy of ϕ is then defined exactly as above.

For two smooth/model metrics ϕ, ψ we have

$$E(\phi)-E(\psi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_{X^{\mathrm{an}}}(\phi-\psi)(dd^c\phi)^j\wedge(dd^c\psi)^{n-j}. \tag{7.2}$$

This is proved using integration by parts in the complex case and follows from basic intersection theory in the non-Archimedean case.

We can draw two main conclusions from (7.2). First, the derivative of the energy functional is the Monge-Ampère operator, in the sense that

$$\frac{d}{dt}E(\phi+tf)\Big|_{t=0} = \int_{X^{\mathrm{an}}} f\,\mathrm{MA}(\phi) \tag{7.3}$$

for a smooth/model metric ϕ on L^{an} and a smooth/model function f on X^{an}.

Second, $E(\psi) \ge E(\phi)$ when $\psi \ge \phi$ are semipositive. It then makes sense to set

$$E(\phi) := \inf\{E(\psi) \mid \psi \ge \phi,\ \psi \text{ a semipositive smooth/model metric on } L^{\mathrm{an}}\}.$$

for any singular semipositive metric $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$. The resulting functional

$$E : \mathrm{PSH}(L^{\mathrm{an}}) \to [-\infty, \infty)$$

has many good properties: E is concave, monotonous, and satisfies $E(\phi + c) = E(\phi) + c$ for $c \in \mathbf{R}$. Further, E is usc and continuous along decreasing nets.

The energy functional singles out a class $\mathcal{E}^1(L^{\mathrm{an}})$ of metrics with *finite energy*, $E(\phi) > -\infty$. This class has good properties. In particular, one can (with some effort) define mixed Monge-Ampère measures $(dd^c\phi)^j \wedge (dd^c\psi)^{n-j}$ for $\phi, \psi \in \mathcal{E}^1(L^{\mathrm{an}})$, and (7.1) continues to hold.

Let us now go back to the variational approach to solving the Monge-Ampère equation. Fix a positive measure μ on X^{an} of mass (L^n). In the complex case we assume μ is absolutely continuous with respect to Lebesgue measure, with density in L^p for some $p > 1$. In the non-Archimedean case we assume that μ is supported on some dual complex. In both cases, one can show that the functional $\phi \to \int(\phi - \phi_0)\,\mu$ is finite and continuous on $\mathrm{PSH}(L^{\mathrm{an}})$, where ϕ_0 is the same reference metric as in (7.1). Thus the functional $F_\mu\colon\ \mathrm{PSH}(L^{\mathrm{an}}) \to [-\infty, \infty)$ defined by

$$F_\mu(\phi) := E(\phi) - \int (\phi - \phi_0)\mu$$

is upper semicontinuous. It follows from (7.2) that F_μ does not depend on the choice of reference metric ϕ_0. We also have $F_\mu(\phi + c) = F_\mu(\phi)$ for $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$, $c \in \mathbf{R}$. Thus F_μ descends to an usc functional on the quotient space $\mathrm{PSH}(L^{\mathrm{an}})/\mathbf{R}$. By Theorem 6.1, the latter space is compact, so we can find $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ maximizing F_μ. It is clear that $\phi \in \mathcal{E}^1(L^{\mathrm{an}})$, so the mixed Monge-Ampère measures of ϕ and ϕ_0 are well defined. However, equation (7.3) no longer makes sense, since there is no reason for the metric $\phi + tf$ to be semipositive for $t \neq 0$. Therefore, it is not clear that $\mathrm{MA}(\phi) = \mu$, as desired. In the next section, we explain how to get around this problem.

8. Envelopes, differentiability and orthogonality

We define the *psh envelope* of a (possibly singular) metric ψ on L^{an} by

$$P(\psi) := \sup\{\phi \in \mathrm{PSH}(L^{\mathrm{an}}) \mid \phi \le \psi\}^*.$$

As before, ϕ^* denotes the usc regularization of a singular metric ϕ. In all cases we need to consider, ψ will be the sum of a metric in $\mathcal{E}^1(L^{\mathrm{an}})$ and a continuous function on X^{an}. In particular, ψ is usc, $P(\psi) \in \mathcal{E}^1(L^{\mathrm{an}})$ and $P(\psi) \le \psi$.

This envelope construction was in fact already mentioned at the end of §6 as it plays a key role in the regularization theorem. The psh envelope is an analogue of the convex hull; see Figure 1.

The key fact about the psh envelope is that the composition $E \circ P$ is differentiable and that $(E \circ P)' = E' \circ P$. More precisely, we have:

Theorem 8.1. *For any $\phi \in \mathcal{E}^1(L^{\mathrm{an}})$ and $f \in C^0(X^{\mathrm{an}})$, the function $t \mapsto E(P(\phi + tf))$ is differentiable at $t = 0$, with derivative $\frac{d}{dt}E(\phi + tf)|_{t=0} = \int f\,\mathrm{MA}(\phi)$.*

Granted this result, let us show how to solve the Monge-Ampère equation. Pick $\phi \in \mathcal{E}^1(L^{\mathrm{an}})$ that maximizes $F_\mu(\phi) = E(\phi) - \int(\phi - \phi_0)\mu$ and consider any $f \in$

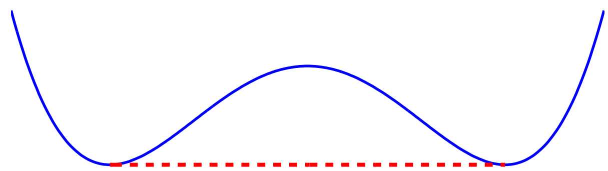

FIGURE 1. The convex hull $P(f)$ of a continuous function f of one variable. Note that $P(f)$ is affine, i.e. $P(f)'' = 0$ where $P(f) \neq f$.

$C^0(X^{\mathrm{an}})$. For any $t \in \mathbf{R}$ we have

$$\begin{aligned} E(P(\phi + tf)) - \int (\phi + tf - \phi_0)\mu &\leq E(P(\phi + tf)) - \int (P(\phi + tf) - \phi_0)\mu \\ &\leq E(\phi) - \int (\phi - \phi_0)\mu. \end{aligned}$$

Since the left hand side is differentiable at $t = 0$, the derivative must be zero, which amounts to $\int f \, \mathrm{MA}(\phi) - \int f\mu = 0$. Since $f \in C^0(X^{\mathrm{an}})$ was arbitrary, this means that $\mathrm{MA}(\phi) = \mu$, as desired.

The proof of this differentiability results proceeds by first reducing to the case when ϕ and f are continuous. A key ingredient is then

Theorem 8.2. *For any continuous metric ϕ on L^{an} we have*

$$\int_{X^{\mathrm{an}}} (\phi - P(\phi)) \, \mathrm{MA}(P(\phi)) = 0. \tag{8.1}$$

In other words, the Monge-Ampère measure $\mathrm{MA}(P(\phi))$ is supported on the locus $P(\phi) = \phi$. A version of this for functions of one variable is illustrated in Figure 1.

To prove this result, we can reduce to the case when ϕ is a smooth/model metric. In the complex case, Theorem 8.2 was proved by Berman and the first author in [BB10] using the pluripotential theoretic technique known as "balayage". In the non-Archimedean setting, Theorem 8.2 is deduced in [BFJ14] from the asymptotic othogonality of Zariski decompositions in [BDPP13] and is for this reason called the *orthogonality property*. The result in [BDPP13] is written in the context of projective varieties over a field. The assumption in Theorem 4.1 that the variety X be defined over a smooth k-curve is used exactly in order to apply the result from [BDPP13].

REFERENCES

[Ale38] A. D. Alexandrov. *On the theory of mixed volumes of convex bodies III: Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies.* (Russian) Mat. Sbornik **3** (1938), no 1, 27-44. [English translation available in Selected works. Part I. Selected scientific papers. Gordon and Breach, Amsterdam, 1996.]

[BR10] M. Baker, R. Rumely. *Potential theory on the Berkovich projective line.* Mathematical Surveys and Monographs, 159. American Mathematical Society, Providence, RI, 2010.

[Ber90] V.G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields.* Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI (1990).

[BB10] R. Berman, S. Boucksom. *Growth of balls of holomorphic sections and energy at equilibrium.* Invent. Math. **181** (2010), 337–394.

[BBGZ13] R. Berman, S. Boucksom, V. Guedj and A. Zeriahi. *A variational approach to complex Monge-Ampère equations.* Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245.

[BT82] E. Bedford and B. A. Taylor. *A new capacity for plurisubharmonic functions.* Acta Math. **149** (1982), 1–40.

[BT87] E. Bedford and B. A. Taylor. *Fine topology, Shilov boundary and $(dd^c)^n$.* J. Funct. Anal. **72** (1987), 225–251.

[BGS95] S. Bloch, H. Gillet and C. Soulé. *Non-Archimedean Arakelov Geometry.* J. Algebraic Geom. **4** (1995), 427–485.

[Bło03] Z. Błocki *Uniqueness and stability for the complex Monge-Ampère equation on compact Kähler manifolds.* Indiana Univ. Math. J. **52** (2003), 1697–1701.

[BK07] Z. Błocki and S. Kołodziej. *On regularization of plurisubharmonic functions on manifolds.* Proc. Amer. Math. Soc. **135** (2007), 2089-2093.

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun and T. Peternell. *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* J. Algebraic Geom. **22** (2013) 201–248.

[BFJ12] S. Boucksom, C. Favre and M. Jonsson. *Singular semipositive metrics in non-Archimedean geometry.* To appear in J. Algebraic Geom. `arXiv:1201.0187`.

[BFJ14] S. Boucksom, C. Favre and M. Jonsson. *Solution to a non-Archimedean Monge-Ampère equation.* J. Amer. Math. Soc., electronically published on May 22, 2014.

[Ceg98] U. Cegrell. *Pluricomplex energy.* Acta Math. **180** (1998), 187–217.

[CL06] A. Chambert-Loir. *Mesures et équidistribution sur les espaces de Berkovich.* J. Reine Angew. Math. **595** (2006), 215–235.

[CL11] A. Chambert-Loir. *Heights and measures on analytic spaces. A survey of recent results, and some remarks.* In *Motivic integration and its interactions with model theory and non-Archimedean geometry.* Volume II. London Mathematical Society Lecture Note Series, 384. Cambridge University Press, Cambridge, 2011.

[CD12] A. Chambert-Loir and A. Ducros. *Formes différentielles réelles et courants sur les espaces de Berkovich.* `arXiv:1204.6277`.

[Dem92] J.-P. Demailly. *Regularization of closed positive currents and intersection theory.* J. Algebraic Geom. **1** (1992), no. 3, 361–409.

[Din09] S. Dinew. *Uniqueness in $\mathcal{E}(X,\omega)$.* J. Funct. Anal. **256** (2009), 2113–2122.

[FJ04] C. Favre and M. Jonsson. *The valuative tree.* Lecture Notes in Mathematics, 1853. Springer-Verlag, Berlin, 2004.

[Gub98] W. Gubler *Local heights of subvarieties over non-archimedean fields.* J. reine angew. Math. **498** (1998), 61–113.

[GZ05] V. Guedj and A. Zeriahi. *Intrinsic capacities on compact Kähler manifolds.* J. Geom. Anal. **15** (2005), 607–639.

[GZ07] V. Guedj and A. Zeriahi. *The weighted Monge-Ampère energy of quasiplurisubharmonic functions.* J. Funct. Anal. **250** (2007), 442–482.

[Koł98] S. Kołodziej. *The complex Monge-Ampère equation.* Acta Math. **180** (1998), 69–117.

[Koł03] S. Kołodziej. *The Monge-Ampère equation on compact Kähler manifolds.* Indiana Univ. Math. J. **52** (2003), 667–686.

[KT00] M. Kontsevich and Y. Tschinkel. *Non-Archimedean Kähler geometry.* Unpublished.

[Lel85] P. Lelong. *Les objets souples de l'analyse complexe.* Exposition. Math. **3** (1985), 149–164.

[Liu11] Y. Liu. *A non-Archimedean analogue of Calabi-Yau theorem for totally degenerate abelian varieties.* J. Diff. Geom. **89** (2011), 87–110.

[RT77] J. Rauch, and B. A. Taylor. *The Dirichlet problem for the multidimensional Monge-Ampère equation.* Rocky Mountain J. Math. **7** (1977), no. 2, 345–364.

[Thu05] A. Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov.* Ph.D. Thesis (2005).

[Yau78] S. T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation.* Comm. Pure Appl. Math. **31** (1978), 339–411.

[YZ13] X. Yuan and S.-W Zhang. *The arithmetic Hodge index theorem for adelic line bundles I.* `arXiv:1304.3538`.

[Zha95] S.-W Zhang. *Positive line bundles on arithmetic varieties.* J. Amer. Math. Soc. **8** (1995), 187–221.

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