EXCLUDED HOMEOMORPHISM TYPES FOR DUAL COMPLEXES OF SURFACES

DUSTIN CARTWRIGHT

Abstract. We study an obstruction to prescribing the dual complex of a strict semistable degeneration of an algebraic surface. In particular, we show that if Δ is a complex homeomorphic to a 2-dimensional manifold with negative Euler characteristic, then Δ is not the dual complex of any semistable degeneration. In fact, our theorem is somewhat more general and applies to some complexes homotopy equivalent to such a manifold. Our obstruction is provided by the theory of tropical complexes.

The dual complex of a semistable degeneration is a combinatorial encoding of the combinatorics of the components of the special fiber. In recent years, it has been studied because of connections to tropical geometry [HK12], non-Archimedean analytic geometry [Ber99], and birational geometry [dFKX12, BF14]. In this paper, we study obstructions to realizing arbitrary complexes as dual complexes of degenerations of surfaces.

We let R be any rank 1 valuation ring and we will consider a *degeneration* over R to be a flat, proper scheme $\mathfrak{X}$ over $\operatorname{Spec} R$ which is strictly semistable in the sense of [GRW14, Sec. 3]. The dual complex of $\mathfrak{X}$ is a Δ-complex with one vertex for each component of the special fiber and higher-dimensional simplices for each connected component where components intersect.

Since semistability implies that the special fiber is normal crossing, the dimension of the dual complex is at most the relative dimension of the family $\mathfrak{X}$. In dimension 1, any graph is the dual complex of some degeneration of curves [Bak08, Cor. B.3]. However, in this paper, we show that the analogous statement is not true in dimension 2.

Theorem 1. *There is no strict semistable degeneration of surfaces over a rank* 1 *valuation ring R such that the dual complex of the special fiber is homeomorphic to a topological surface Σ with $\chi(\Sigma) < 0$.*

We conjecture that Theorem 1 can be strengthened to replace "homeomorphic" with "homotopy equivalent." In fact, we can prove a strengthening in this direction which applies to Δ-complexes formed from a manifold with negative Euler characteristic by attaching additional simplices in a controlled way. First, what we call "fins" are allowed so long as they don't change the homotopy type and where the gluing is along a subset that's not too complicated. Second, arbitrary complexes may be attached to the manifold, so long as the gluing is along a finite set. These complexes are collectively the "ornaments" in the following definition.

Definition 2. We say that a 2-dimensional Δ-complex Δ is a *manifold with fins and ornaments* if there exists a subcomplex Σ, the *manifold*, subcomplexes $F_1, \ldots, F_n$, the *fins*, and a subcomplex O, the *ornaments*, such that:

(1) We have a decomposition $\Delta = \Sigma \cup F_1 \cup \ldots \cup F_n \cup O$.
(2) Σ is homeomorphic to a 2-dimensional topological manifold.
(3) For any i, F_i is contractible and $F_i \cap \Sigma$ is a path.
(4) For any $i > j$, $F_j \cap F_i$ is a subset of the endpoints of the path $F_i \cap \Sigma$.
(5) The intersection $O \cap (\Sigma \cup F_1 \cup \cdots F_n)$ is a finite set of points.

If the manifold Σ has negative Euler characteristic, we call Δ a *hyperbolic manifold with fins and ornaments* and if the subcomplex O is empty, then we call Δ a *manifold with fins*.

Theorem 3. *If Δ is a hyperbolic manifold with fins and ornaments, then there is no degeneration with dual complex Δ.*

The obstruction to having the dual complex of a degeneration be a hyperbolic manifold with fins and ornaments is in lifting the dual complex to a tropical complex. A tropical complex is Δ-complex, together with the intersection numbers of the 1-dimensional strata inside the 2-dimensional strata, which are called the structure constants of the tropical complex [Car13].

Theorem 4. *If Δ is a hyperbolic manifold with fins and ornaments, then there is no tropical complex with Δ as its underlying topological space.*

There are two characteristics of the special fiber of a degeneration which are incorporated into the axioms of a tropical complex. The first is that the special fiber is principal which gives a relationship among the intersection numbers with a fixed curve. The second is that Hodge index theorem, which restricts the possible intersection matrices of a fixed surface in the special fiber.

Both axioms of a tropical complex are necessary in the proof of the obstruction. Without the condition coming from the Hodge index theorem, the object would only be a weak tropical complex, and any Δ-complex lifts to a weak tropical complex. For example, if Δ is homeomorphic to a topological manifold, then choosing all structure constants equal to 1 gives a weak tropical complex, but this will not be a tropical complex if $\chi(\Delta) < 0$.

On the other hand, Kollár has shown that any finite n-dimensional simplicial complex is realizable as the dual complex of a simple normal crossing divisor [Kol14, Thm. 1], but such a divisor would not give a tropical complex because the divisor is not necessarily principal. However, when connected, such a divisor can be realized as the exceptional locus of the resolution an normal, isolated singularity [Kol14, Thm. 2]. Thus, we see Theorem 3 as an example of how the global geometry of a smooth algebraic variety is more restricted than the local geometry of a singularity, in line with [KK14].

We also note that unlike the cases in Theorem 1, topological surfaces with non-negative Euler characteristic are all possible as homeomorphism types of degenerations. In particular, the 2-sphere, the real projective plane, the torus, and the Klein bottle appear as degenerations of K3 surfaces, Enriques surfaces, Abelian surfaces, and bielliptic surfaces respectively. In fact, a partial converse is possible in that the dual complexes of such degenerations have been classified by results of Kulikov, Persson, Pinkham, and Morrison [Kul77, Per77, PP81, Mor81]. Note that topological surfaces of non-negative Euler characteristic all arose from degenerations of varieties of Kodaira dimension 0. However, these classification results would already suffice to prove Theorem 1 if we assumed that the general fiber had Kodaira dimension 0.

1. Tropical complexes

We begin by recalling some of the basic properties of tropical complexes, as introduced in [Car13]. In short, a tropical complex is a Δ-complex, augmented with integers, called the *structure constants*, and satisfying certain hypotheses [Car13, Def. 2.1] In this paper, we will deal exclusively with 2-dimensional tropical complexes, which we will denote *tropical surfaces*.

The primary importance of the structure constants for us will be that it leads to a sheaf $\mathcal{A}$ of affine linear functions on a tropical surface. On the interior of each simplex, a local section of $\mathcal{A}$ is exactly an $\mathbb{R}$-valued affine linear function with integral slopes, but the definition more generally depends on the structure constants. All constant functions are affine linear, and if we denote by $\mathcal{D}$ the quotient $\mathcal{A}/\mathbb{R}$, where $\mathbb{R}$ is the sheaf of locally constant $\mathbb{R}$-valued functions, then we have a long exact sequence in sheaf cohomology [Car15, Sec. 3]:

$$0 \to H^0(\Delta, \mathbb{R}) \to H^0(\Delta, \mathcal{A}) \to H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{R}) \to \cdots \tag{1}$$

One of the main results from [Car15] is the following:

Proposition 5 (Prop. 4.4 in [Car15])**.** *If Δ is a tropical surface which is locally connected through codimension 1, the $\mathbb{R}$-span of the image of the morphism $H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{R})$ has codimension at most 1 in $H^1(\Delta, \mathbb{R})$.*

In Proposition 5, *locally connected through codimension* 1 means that the link of each vertex is connected.

In [Car13, Sec. 2], tropical complexes are constructed from regular semistable degenerations over discrete valuation rings. Although we work over a possibly non-discrete valuation ring, all the data of a tropical complex can be obtained from the special fiber, which is a reduced simple normal crossing scheme over the residue field of R. However, even for regular degenerations over a discrete valuation ring, getting a tropical complex requires an additional technical condition of robustness in dimension 2 [Car13, Prop. 2.7], and without this condition we only get a weak tropical complex. The distinction is that each vertex v of a weak tropical complex Δ has a local intersection matrix M_v, which is always symmetric, but has to have exactly one positive eigenvalue for Δ to be a tropical complex. Analogously, we get weak tropical complexes from degenerations over non-discrete valuation rings.

Proposition 6. *The special fiber of any degeneration $\mathfrak{X}$ yields a weak tropical complex Δ such that the local intersection matrix M_v has at most one positive eigenvalue for each vertex v of Δ.*

Proof. Recall from [Car13, Def. 2.1] or [Car15, Def. 2.1] that in order for the dual complex Δ with structure constants from the intersection numbers to be a weak tropical complex, we need that for any edge e of Δ, with endpoints v and w, the structure coefficients satisfy the equality:

$$\alpha(v, e) + \alpha(w, e) = \deg(e). \tag{2}$$

Let C_e be the curve corresponding to e in the special fiber of $\mathfrak{X}$, and by our semistability condition, on a Zariski open neighborhood meeting C_e, there is an étale map to $\operatorname{Spec} R[x, y, z]/\langle xy - \pi\rangle$ for some element π in the maximal ideal of R. Then, the principal Cartier divisor defined by π can be written, at least in a neighborhood of C_e, as the union of Cartier divisors, each of which is supported on an irreducible component of the special fiber of $\mathfrak{X}$. For example, in the above chart, the functions x and y pull back to give defining equations for each of the components containing C_e.

Thus, using linearity of the intersection product [Gub03, Prop. 5.9(b)], we can split up the intersection of the principal divisor defined by π with the curve C_e into terms coming from the components of the special fiber of $\mathfrak{X}$. For components of the special fiber which don't contain C_e, if we pull back to C_e we get a Cartier divisor equal to the points of intersection, with multiplicities equal to 1. Thus, the degree of the intersection of such a Cartier divisor with C_e is equal to the number of points of intersection by the projection formula [Gub03, Prop. 5.9(c)]. The total degree for all components which don't contain C_e gives the right-hand side of (2).

Now consider the two components C_v and C_w containing C_e. If we pull back the Cartier divisor supported on C_v to C_w then we get the divisor C_e on C_w. The self-intersection of C_e is $-\alpha(v, e)$ by the definition of the structure constants. Thus, using the projection formula again, the components containing C_e contribute a cycle of degree equal to $-\alpha(v, e) - \alpha(w, e)$, so the equality (2) follows because π obviously defines a principal divisor.

Finally, as in [Car13, Sec. 2], the local intersection matrix M_v records the intersection theory on the surface of the special fiber corresponding to v, restricted to curves of the special fiber. By the Hodge index theorem, this matrix can have at most one positive eigenvalue. □

One approach to obtaining a tropical complex instead of a weak tropical complex is Proposition 2.9 in [Car13], which shows that for degenerations with projective components, robustness can be obtained by appropriate blow-ups. While this proposition could be adapted to the case of non-discrete valuations, while also keeping track of the effect on the underlying topological space, it is more convenient to perform the modification combinatorially:

Lemma 7. *Let Δ be a 2-dimensional weak tropical complex and suppose that for each vertex v of Δ, the local intersection matrix M_v has at most one positive eigenvalue. Then, there exists a tropical surface Δ' such that the underlying topological space of Δ' is formed by attaching a finite number of 2-simplices to edges of Δ.*

Proof. We suppose that v is a vertex of Δ such that M_v has no positive eigenvalues, i.e. it is negative definite. Let e be an edge containing v and let w be the other endpoint of e. We attach an additional 2-simplex onto e and label the new vertex u', with the new edges e'_v and e'_w. We use v', w' and e' to denote the representatives of v, w, and e in the new weak tropical complex Δ'. We assign the coefficients on Δ' to be the same as on Δ, except that:

$$\begin{aligned} \alpha(w', e') &= \alpha(w, e) & \alpha(v', e') &= \alpha(v, e) + 1 \\ \alpha(w', e'_w) &= 0 & \alpha(u', e'_w) &= 1 \\ \alpha(v', e'_v) &= 2 & \alpha(u', e'_v) &= -1 \end{aligned}$$

Then one can check that $M_{v'}$ has one more positive eigenvalue than M_v, $M_{w'}$ has the same number as M_w and $M_{u'}$ has exactly one positive eigenvalue. By repeating this process, we can construct the desired tropical complex Δ'. □

2. Proof of the main theorems

The crux of Theorem 4 and thus of Theorem 3 is the following lemma:

Lemma 8. *Let Δ be a tropical surface whose underlying Δ-complex is a manifold with fins and is connected through codimension 1. If s is a 2-dimensional simplex contained in the manifold subcomplex of Δ, and U_s denotes the interior of s, then the restriction map*

$$H^0(\Delta, \mathcal{D}) \to H^0(U_s, \mathcal{D}) \cong \mathbb{Z}^2$$

is injective.

Proof. Note that the isomorphism $H^0(U_s, \mathcal{D}) \cong \mathbb{Z}^2$ holds because affine linear functions on U_s are equivalent to affine linear functions with integral slopes on a standard simplex in $\mathbb{R}^2$. Thus, $\mathcal{A}$ restricted to U_s is isomorphic to the locally constant sheaf with values in $\mathbb{R} \times \mathbb{Z}^2$, and the quotient sheaf $\mathcal{D} = \mathcal{A}/\mathbb{R}$ is isomorphic to $\mathbb{Z}^2$.

Now, we let Σ and $F_1, \ldots, F_n$ denote the manifold and fins of the simplicial complex, as in Definition 2. We suppose ω a global section of $\mathcal{D}$ such that the restriction of ω to U_s is trivial, and we want to show that ω is trivial. We start with $V = U_s$ and then we'll expand the open set V until it is all of Δ. At each step, V will either be disjoint from, or contain, each fin F_i, so the boundary of V will be contained in Σ.

First suppose that there exists an edge e in the boundary of V such that e is not contained in any of the fins. Let f denote the 2-simplex bordering e whose interior is in V and let f' denote the 2-simplex on the other side of e.

Then, the construction of affine linear functions on a tropical complex [Car13, Constr. 4.2] identifies the union of the interiors of f, f', and e with an open subset of $\mathbb{R}^2$, and the sections of $\mathcal{A}$ are exactly affine linear functions with integral slope on this set. As above, $\mathcal{A}$ and $\mathcal{D}$ are therefore locally constant sheaves with values in $\mathbb{R} \times \mathbb{Z}^2$ and $\mathbb{Z}^2$ respectively. Thus, we can expand V to include the interiors of e and f', where ω is also zero.

Second, we assume that every edge in the boundary of V is contained in some $\Sigma \cap F_i$. Let i be the maximal index such that F_i intersects the boundary of V. Then, the entire path $\Sigma \cap F_i$ must be in the boundary of V or else there would be a fin F_j with $j < i$ intersecting F_i not at its endpoint, which would contradict Definition 2. In particular, ω must be constant along $\Sigma \cap F_i$.

Let $\widetilde{\mathcal{A}}_i$ be the sheaf of piecewise linear functions on Δ whose associated divisors are contained in $\Sigma \cap F_i$. If we let $\widetilde{\mathcal{D}}_i$ denote the quotient sheaf $\widetilde{\mathcal{A}}_i/\mathbb{R}$ on Δ, then we can construct a global section $\widetilde{\omega}_i$ of $\widetilde{\mathcal{D}}_i$ which is equal to ω on F_i, but identically zero away from F_i. Consider the long exact sequence of cohomology associated to the quotient $\widetilde{\mathcal{D}}_i$, analogous to (1):

$$0 \to H^0(\Delta, \mathbb{R}) \to H^0(\Delta, \widetilde{\mathcal{A}}_i) \to H^0(\Delta, \widetilde{\mathcal{D}}_i) \to H^1(\Delta, \mathbb{R}) \to$$

Since $\widetilde{\omega}_i$ is only non-trivial on F_i, which is contractible, the image of $\widetilde{\omega}_i$ in $H^1(\Delta, \mathbb{R})$ is trivial, so $\widetilde{\omega}_i$ lifts to an element of $H^0(\Delta, \widetilde{\mathcal{A}}_i)$, which we also denote by $\widetilde{\omega}_i$ and we choose the representative such that $\widetilde{\omega}_i$ is zero on Σ.

If $\widetilde{\omega}_i$ is non-constant, then it must have a minimum value strictly less than zero or maximum value strictly greater than zero. Then, it would have its minimum or maximum, respectively, on $F_i \setminus \Sigma$. In either case, we apply Proposition 2.8 from [Car15], which implies that the divisor of $\widetilde{\omega}_i$ is non-trivial in a neighborhood of where it achieves its minimum or maximum. This contradicts our definition of a section of the sheaf $\widetilde{\mathcal{A}}_i$, so we conclude that $\widetilde{\omega}_i$ is identically zero. Thus, ω is identically zero on F_i, so we can expand V to include $F_i \setminus \Sigma$.

We've now shown that for each edge e of $\Sigma \cap F_i$, the section ω is zero on all but one simplex containing e, namely the simplex in Σ on the other side from V. By the condition of being the quotient of an affine linear function means that ω must also be zero on this simplex. Thus, we further expand V to also include the interiors of all 2-simplices containing $\Sigma \cap F_i$.

At the end, we will have that ω is zero on an open set V which contains the interior of every 2-simplex in Δ and since affine linear functions are continuous by definition, this means that ω is zero, which finishes the proof of the lemma. □

We use Lemma 8 to prove the following strengthening of Theorem 4.

Theorem 9. *If Δ is a hyperbolic manifold with fins and ornaments, then there is no weak tropical complex, with Δ as its underlying topological space, and such that for every vertex v of Δ, M_v has at most one positive eigenvalue.*

Proof. Suppose that Δ is a weak tropical surface whose underlying Δ-complex is as in the theorem statement. We can assume that when decomposing Δ as in Definition 2, the subcomplex of ornaments O is maximal, so that if we let Δ' denote the subcomplex consisting of just the manifold and fins, then Δ' is locally connected through codimension 1. Then, taking the restriction of the structure constants from Δ, we get that Δ' has the structure of a weak tropical complex, because the local structure around each edge of Δ' is unchanged. Moreover, at each vertex v of Δ', the local intersection matrix M'_v is a submatrix of the corresponding matrix M_v for Δ. Therefore, M'_v also has at most one positive eigenvalue.

Next, we apply Lemma 7 to transform Δ' into a tropical surface Δ'' by gluing simplices onto edges of Δ'. Whenever we glue a simplex onto an edge e which is contained in one of the fins $F_i \subset \Delta'$, we can include that simplex in the fin, which remains contractible and its intersection with the manifold Σ is unchanged. If we glue a simplex onto an edge e contained in the manifold Σ, then the simplex forms a new fin F_{n+1}, numbered after all the other fins. Since $F_{n+1} \cap \Sigma$ is a single edge, the intersection of F_{n+1} with any other fin will be a subset of the endpoints of this edge. Thus, Δ'' is still a hyperbolic manifold with fins. Finally, if the manifold $\Sigma \subset \Delta''$ is not orientable, we can replace Δ'' with its oriented cover of Δ'' so that $\chi(\Delta'') \leq -2$.

Now we consider the cohomology group $H^1(\Delta'', \mathcal{D})$ to get our contradiction. By Lemma 8, $H^0(\Delta'', \mathcal{D})$ is a subgroup of $\mathbb{Z}^2$, so it is a free Abelian group of rank at most 2. On the other hand, if Δ'' is homotopy equivalent to a manifold Σ with $\chi(\Delta'') \leq -2$, then

$$\dim_{\mathbb{R}} H^1(\Delta'', \mathbb{R}) = 2 - \chi(\Delta'') \geq 4.$$

Moreover, by Proposition 5, the $\mathbb{R}$-span of the image of $H^1(\Delta'', \mathcal{D})$ has codimension at most 1 in $H^1(\Delta'', \mathbb{R})$, so the rank of $H^1(\Delta'', \mathcal{D})$ is at least $\dim_{\mathbb{R}} H^1(\Delta'', \mathbb{R}) - 1 \geq 3$. Thus, we have a contradiction regarding the rank of $H^1(\Delta'', \mathcal{D})$, so the weak tropical surface Δ cannot exist. □

Proof of Theorem 3. Suppose $\mathfrak{X}$ is a strict semistable degeneration whose dual complex Δ is a hyperbolic manifold with fins and ornaments. Then, by Proposition 6, Δ has the structure of a weak tropical complex such that the matrix M_v has at most one positive eigenvalue for every vertex v. However, by Theorem 9, such a weak tropical complex cannot exist, so we conclude that $\mathfrak{X}$ cannot exist. □

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Department of Mathematics, University of Tennessee, 227 Ayres Hall, Knoxville, TN 37996-1320

E-mail address: `cartwright@utk.edu`