Piecewise monomial skeleta

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1 Introduction

The purpose of this text is to describe roughly a joint work with Amaury Thuillier, which is still in progress. It consists of a systematic study of the subsets of Berkovich spaces that "locally look like finite polyhedral complexes". It is partially motivated by Thuillier's current work on the homotopy type of Berkovich spaces, for which a better understanding of the behavior of those subsets has turned out to be necessary.

Let us give some precisions about the objects involved; all analytic spaces we consider are defined over a fixed ground field k. Following Berkovich, we will use the *multiplicative* notation: in what follows, a *polyhedron* will be a subset of $(\mathbb{R}_+^\times)^n$ isomorphic through the logarithm $(\mathbb{R}_+^\times)^n \simeq \mathbb{R}^n$ to a finite union of simplices with *rational* slopes. A map between two such polyhedra will be said to be PL (resp. $\mathbb{Z}$-PL) if it is continuous and piecewise affine with rational (resp. integral) linear parts.

One can take polyhedral complexes as building blocks for defining the category of abstract PL spaces (resp. $\mathbb{Z}$-PL spaces). Roughly speaking, those spaces are defined using polyhedral atlases, consisting of coverings by polyhedral charts with PL (resp. $\mathbb{Z}$-PL) transition isomorphisms, but the precise construction is slightly technical, because those atlases consist of compact (and usually not open) pieces; the interested reader may find details in [2], see also [5].

Now let us say that a locally closed subset S of an analytic space X is a *skeleton* if there exists a PL-structure on S having the following properties.

1) For every analytic domain Y of X and every invertible function f on Y, the intersection $S \cap Y$ is a PL-subspace of P and the restriction of $|f$ to $S \cap Y$ is PL.

2) There exists a polyhedral atlas (P_i) on S and, for every i, an analytic domain X_i of X containing P_i and invertible functions $f_{i1}, \ldots, f_{in_i}$ on X_i such that $(|f_{ij}|)_j$ induces a PL-isomorphism between P_i and a polyhedron of $(\mathbb{R}_+^\times)^{n_i}$.

It can easily be shown that such a structure is necessarily unique. Hence a skeleton of X inherits a *canonical* PL-structure, which by definition can be described purely in terms of the analytic structure of X.

The archetypal example of a skeleton is the subset $S_n := \{\eta_r\}_{r \in (\mathbb{R}_+^\times)^n}$ of $\mathbb{G}_m^{n,\mathrm{an}}$, where η_r is for every r the point defined by the semi-norm

$$\sum a_I T^I \mapsto \max |a_I| \cdot r^I.$$

Let us give another example. Let $\mathfrak{X}$ be a polystable formal scheme over k°. The combinatorics of the singularities of the special fiber of $\mathfrak{X}$ can be encoded in an abstract polyhedron, which has been proved by Berkovich in [2] to be naturally PL-isomorphic to a skeleton $S(\mathfrak{X})$ of $\mathfrak{X}_\eta$ (he has moreover established that $\mathfrak{X}_\eta$ admits a deformation retraction to $S(\mathfrak{X})$).

The author has proved ([3], [5]) the following theorem.

Theorem 1. *Let $n \in \mathbb{N}$, let X be a k-analytic space of dimension $\leqslant n$, and let $\varphi_1, \ldots, \varphi_m$ be morphisms from X to $\mathbb{G}_m^n$. Then the union $\bigcup_i \varphi_i^{-1}(S_n)$ is a skeleton; for every i, the map $\varphi_i^{-1}(S_n) \to S_n$ is a piecewise-immersion of PL-spaces.*

Let us say a few words about the proof; its core consists in proving the result when $m = 1$ (one uses some trick to reduce to this case). One can then either use de Jong's alteration in the spirit of Berkovich's work ([1], [2]), which was done in [3]. Or, like in [5], one can use model-theoretic arguments (based upon Hrushovski and Loeser's work [7]) to establish the following result, from which theorem 1 follows through standard algebraization procedures: *let k be* any *valued field, and let L be a finite extension of $k(T_1, \ldots, T_n)$; there exists a finite subset of L that separates the extensions of* any *Gauss valuation on* $k(T_1, \ldots, T_n)$. Here a Gauss valuation is a valuation given of the form $\sum a_I T^I \mapsto \max |a_I| \cdot r^I$ for r an n-uple of elements of an abelian ordered group containing $|k^\times|$.

Let X be a k-analytic space of dimension n, and let (X_i) be a G-covering of X by analytic domains. For every i, les $(\varphi_{ij})_j$ be a finite family of morphisms from X_i to $\mathbb{G}_m^{n,\mathrm{an}}$, and set $\Sigma_i = \bigcup_j \varphi_{ij}^{-1}(S_n)$ (it is a skeleton by theorem 1). If S is a subset of X such that $S \cap \Sigma_i$ is a PL-subspace of Σ_i for every i, then S is easily seen to be a skeleton. Such a skeleton will be called *nice* in the sequel.

2 Integral structures on, and direct images of, skeleta

In our work in progress with Thuillier, we have established the following results (their proofs have been completely written down).

Theorem 2. *Any nice skeleton inherits a natural* $\mathbb{Z}$-PL *structure.*

Theorem 3. *Let n and d be two integers, let Y be an n-dimensional analytic space, let X be a d-dimensional analytic space, and let $\varphi\colon Y \to X$ be a compact morphism of pure dimension $n - d$. Then if S is a nice skeleton of Y, its image $\varphi(S)$ is a nice skeleton of X.*

Let us quickly mention the main ingredients of the proofs. For theorem 2, we use a weak desingularisation theorem by Knaf and Kuhlmann [10], namely the local uniformisation of Abhyankar valuations on algebraic varieties, whic we apply at the residue field level. This allows us to reduce to the case of the skeleton S_n of $\mathbb{G}_m^{n,\mathrm{an}}$ whose integral structure is the obvious one.

As far as theorem 3 is concerned, its proof is essentially based upon a description of germs of nice skeleta at a given point x of an analytic space X, in terms of Temkin's graded reduction $\widetilde{(X, x)}$: such a germ is given by a quasi-compact subspace of $\widetilde{(X,x)}$ consisting of "monomial valuations". The core of the work is the precise definition of those "residual skeleta" and the construction of the dictionary between germs of nice skeleta and residual skeleta, which uses Temkin's theory and quite elementary model theory, applied to the theory of divisible abelian ordered group (a compact nice skeleton can be seen as a definable object in this theory). The proof of th. 3 is thereafter slightly formal: one proves using basic facts from valuation-theory together with (quasi)-compactness arguments that for every $y \in S$ the image of the residual skeleton $\widetilde{(S,y)}$ is a residual skeleton of $\widetilde{(X,\varphi(y))}$; then our dictionary allows to lift this assertion at the level of germs of nice skeleta, hence to conclude because the property of being a nice skeleton is local.

3 The case of piecewise-monomial skeleta

Our current goal (which is not yet completely achieved) is now to extend those results to the case of *piecewise-monomial* skeleta. Let us first quickly explain what those objects are.

By its very definition, a subset of $(\mathbb{R}_+^\times)^n$ is a polyhedron if it can be defined by finitely many non-strict inequalities between monomial functions with integral exponents, which can always be taken non-negative. But such a condition also makes sense for a subset of $\mathbb{R}_+^n$, and can thus be used, in a way very similar to what we followed concerning PL spaces, to define the categories of PM and $\mathbb{Z}$-PM spaces (PM stands for *piecewise monomial*); note that the multiplicative notation is much more convenient than the additive one (which would require to add $+\infty$ or $-\infty$ to $\mathbb{R}$) for such a purpose.

We can then define a PM-skeleta of a Berkovich space, as we did for skeleta. The archetypal example of a PM-skeleton is the subset S'_n of $\mathbb{A}^{n,\mathrm{an}}$ that consists of all points of the form η_r for $r \in \mathbb{R}_+^n$. We have proved the following analogue of theorem 2.

Theorem 4. *Let X be an analytic space and let $(\varphi_i\colon X \to \mathbb{A}^{n_i,\mathrm{an}})_{1\leqslant i\leqslant m}$ be a family of zero-dimensional morphisms. Then the union $\bigcup \varphi_i^{-1}(S'_{n_i})$ is a* PM*-skeleton. It inherits a natural* $\mathbb{Z}$-PM *structure, and for every i the map $\varphi_i^{-1}(S'_{n_i}) \to S'_{n_i}$ is a piecewise immersion of* PM*-spaces.*

Before saying a few words about its proof, let us mention that we are currently trying to establish an analogue of theorem 3 too, but we face some technical problems and the situation is much more complicated than what we expected (theorem 4 allows to define the notion of a nice PM-skeleton of a Berkovich space, but this notion does not seem to be stable under compact direct images, contrary to what happens for skeleta; nevertheless, it seems plausible that the image of a nice PM-skeleton under a compact map is still a PM-skeleton).

Some words about the proof of theorem 4 and the space $\widehat{X}$

Contrary to what we thought at the very beginning, it seems that we do not need to use theorem 2 in order to prove theorem 4: we have a direct proof of the latter, which will then provide a new proof of theorem 2.

Our proof of theorem 4 is mainly valuation-theoretic. But we deal with piecewise-monomial skeletta, which can meet proper Zariski-closed subsets of the ambiant space (for example, S'_n meets all coordinate hyperplanes, contrary to S_n, all whose points are Zariski-generic on $\mathbb{G}_m^n$). This prevents us algebraizing the situation, and then using tools from algebraic geometry and/or from the model-theory of valued fields. Hence we have to work with valuations on arbitrary equi-characteristic excellent local rings, which requires a lot of commutative algebra and relative algebraic geometry over such a ring.

For instance, we prove that if A is such a ring, which we assume to be a domain, then for every Abhyankar valuation $|.|$ on Spec A centered at the closed point, and supported at the generic point, the valued field (Frac $A, |.|$) is stable (*i.e* every finite extension of it is defectless). Our proof follows the general philosophy outlined for example in [12]: reduction to the height 1 case by some dévissage, completion, and then use of the corresponding result for Abhyankar points of Berkovich spaces – which itself can be easily reduced to the corresponding result for algebraic Abhyankar valuation, which is now classical (Kuhlmann, [11]). This stability result is crucial, and in some sense replaces the aforementioned model-theoretic arguments of [5] (but note that the stability of algebraic Abhyankar valuations can also be proved by similar model-theoretic methods, see [6]).

But of course, looking at what happens for a particular valuation can not be sufficient; one needs at some point more global tameness and/or finiteness result. For that purpose, we have introduced a variant of the Zariski-Riemann space, which we will now describe. Let $X = \mathscr{M}(A)$ be an affinoid space. Following Kedlaya, let us call a *reified valuation* the datum of a valuation on A (with possibly non-trivial kernel) and of an order-preserving embedding of $\mathbb{R}_+^\times$ into its group. There is a natural notion of equivalence of such reified valuations, and we define $\widehat{X}$ as the set of all reified valuation x on A such that $|f(x)| \leqslant ||f||$ for every $f \in A$ (here $||.||$ is the spectral norm). Note than contrary to Kedlaya in *loc. cit*, we do not require those valuations to be continuous. For example, assume that $A = k\{T\}$. The valuation x obtained by composing the vanishing order at the origin and the given absolute value on k belongs to $\widehat{X}$, but is not continuous. The element $|T(x)|$ is infinitely closed to zero (but still positive) with respect to $\mathbb{R}_+^\times$.

Remark. It is essential for our purpose to allow such valuations because we need to have a control on what happens in the neighborhood of a proper Zariski-closed subset of X. For example if X is a nodal curve, then every branch at the singularity will give rise to a point of $\widehat{X}$ (by composing the corresponding discrete valuation with the absolute value of the residue field).

We will now end this short report by simply mentioning some properties of $\widehat{X}$ which play a crucial role in our proof.

Let us say that a subset of X (resp. $\widehat{X}$) is semi-algebraic if it can be defined by a boolean combination of inequalities $|f| \bowtie \lambda|g|$ with f and g in A, with $\lambda \in \mathbb{R}_+$ and with $\bowtie \in \{<, >, \leqslant, \geqslant\}$. We endow $\widehat{X}$ with the topology generated by the semi-algebraic subsets, for which it is easily shown –this is slightly formal – to be compact; its semi-algebraic subsets are then precisely its compact-open subsets, and any ultra-filter of semi-algebraic subsets of $\widehat{X}$ is principal.

There is a natural (not continuous !) embedding $X \hookrightarrow \widehat{X}$, whose image is dense: this is essentially the rephrasing of a result proved by Huber in [8]. This implies that $F \mapsto F \cap X$ induces a bijection between the set of semi-algebraic subsets of $\widehat{X}$ and the set of semi-algebraic subsets of X; hence $\widehat{X}$ can be interpreted as the set of ultra-filters of semi-algebraic subsets of X.

Now let V be an affinoid domain of X. It is semi-algebraic by Gerritzen-Grauert's theorem, hence corresponds to a semi-algebraic subset $V_{\widehat{X}}$ of $\widehat{X}$. On the other hand, the space $\widehat{V}$ is well-defined, and there is a natural continous map $\widehat{V} \to \widehat{X}$. One proves easily (using descriptions in terms of ultra-filters) that its image is precisely $V_{\widehat{X}}$, but *in general this map is not injective* – this the typical example of the problems we face.

It is not difficult to give a counter-example to injectivity: assume that X is the unit bi-disc, that V is a closed disc of biradius $(r, 1)$ with $r < 1$, and that $f \in k\{T/r\}$ has radius of convergence exactly equal to r, and spectral norm $\leqslant 1$. Let Y be the Zariski-closed subset of V defined by the equation $T_2 = f(T_1)$ (it can be identified with the one-dimensional disc of radius r, through the map (Id, f)) ; let η be the unique point of the Shilov boundary of Y. We can see η as belonging to $\widehat{V}$, and its image on $\widehat{X}$ is nothing but η, seen as an element of $X \hookrightarrow \widehat{X}$.

Let η^+ be the composition of the vanishing order along Y and of the valuation η. It belongs to $\widehat{V}$. Let $g \in \mathscr{O}(V)$. If g vanishes along Y then $|g(\eta)^+|$ is infinitely closed to zero (but non-zero if $g \neq 0$, *e.g.* $g = T_2 - f(T_1)$); if not then $|g(\eta^+)| = |g(\eta)|$. It is not difficult to prove that Y is Zariski-dense in X (because the radius of f is exactly r, which prevents Y being extended to curve around η); hence for any function g on X one has $|g(\eta^+)| = |g(\eta)|$. Therefore the image of η^+ on $\widehat{X}$ is also equal to η.

But in fact, as far as the valuation we consider are Abhyankar (and being interested in skeleta, we basically more or less only have to deal with Abhyankar valuations), this problem can not happen. Indeed, we prove that *if a point ξ of $V_{\widehat{X}}$ is Abhyankar, then it has a unique pre-image ζ on $\widehat{V}$; moreover, ζ has the same residue field and the same value group as those of ξ.*

The proof is technically involved; it uses the fact that the map $\mathrm{Spec}\, \mathscr{O}(V) \to \mathrm{Spec}\, \mathscr{O}(X)$ is regular (due to the author, see [4]), and also a result by Raynaud (*cf.* prop. 21.4.9 in the *Errata* of EGA IV.4) giving a necessary and sufficient condition, given a flat morphism $Y \to X$ between noetherian schemes, for a Cartier divisor on Y to be the pull-back of a Cartier divisor on X.

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