Tropical and nonarchimedean analytic geometry of curves

Sam Payne

August 11, 2011

Joint work with

- Matt Baker
- Joe Rabinoff

Tropicalization

- K is a valued field, with valuation $\nu : K^* \to \mathbb{R}$.
- X is a subvariety of $\mathbb{G}_m^n$ over K.
- $\text{Trop}(X) \subset \mathbb{R}^n$ is the set of valuations of points of X.

Tropicalization

- K is a valued field, with valuation $\nu : K^* \to \mathbb{R}$.
- X is a subvariety of $\mathbb{G}_m^n$ over K.
- $\mathrm{Trop}(X) \subset \mathbb{R}^n$ is the set of valuations of points of X.

For $L|K$ an extension of valued fields, and $x = (x_1, \dots, x_n)$ a point with coordinates in L, let

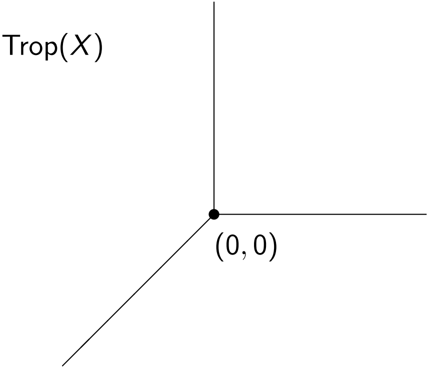
$$\mathrm{Trop}(x) = (\nu(x_1), \dots, \nu(x_n)).$$

Definition

$$\mathrm{Trop}(X) = \{\mathrm{Trop}(x) \mid x \in X(L) \text{ for some } L|K\}.$$

Example: $(x + y + 1 = 0)$

Let $X \subset \mathbb{G}_m^2$ be the "line" cut out by $(x + y + 1) = 0$.



Relation to Gröbner theory

Regular functions on $T = \mathbb{G}_m^n$ are Laurent polynomials:

$$a_1 x^{u_1} + \cdots + a_r x^{u_r}$$

with $a_i \in K$ and $u_i \in \mathbb{Z}^n$.

Definition

The w-weight of a monomial ax^u is

$$\mathrm{wt}(ax^u) = \nu(a) + \langle u, w \rangle.$$

Initial degenerations of T

Let R_w be the ring generated by monomials of nonnegative w-weight.

Definition

The tropical integral model of T at w is

$$\mathcal{T}_w = \mathrm{Spec}(R_w)$$

- Scheme over the valuation ring $R \subset K$ with general fiber T_K.
- Special fiber $\mathrm{in}_w T$ is a torsor over T_k.

Here, $k = R/\mathfrak{m}$ is the residue field.

Initial degenerations of X

Let $\mathcal{X}_w$ be the closure of X in $\mathcal{T}_w$.

Definition

The initial degeneration $\mathrm{in}_w X$ is the special fiber of $\mathcal{X}_w$.

Roughly speaking, $\mathrm{in}_w X$ is cut out by "w-initial forms" of Laurent polynomials in I_X.

Initial degenerations of X

Let $\mathcal{X}_w$ be the closure of X in $\mathcal{T}_w$.

Definition

The initial degeneration $\operatorname{in}_w X$ is the special fiber of $\mathcal{X}_w$.

Roughly speaking, $\operatorname{in}_w X$ is cut out by "w-initial forms" of Laurent polynomials in I_X.

- $\mathcal{X}_w$ is not proper, so the closure of a point in the general fiber may not meet the special fiber.
- If the initial form of some $f \in I_X$ is a monomial then $\operatorname{in}_w X$ is empty.
- The closure of a point $x \in X(K)$ meets $\operatorname{in}_w X$ if and only if $\operatorname{Trop}(x) = w$.

The "Fundamental Theorem"

Theorem

1. $\mathrm{Trop}(X)$ *is the set of* $w \in \mathbb{R}^n$ *such that* $\mathrm{in}_w X$ *is not empty.*
2. *It is the underlying set of a connected polyhedral complex of pure dimension* $\dim X$*.*
3. *This polyhedral complex can be chosen such that all faces have rational slope.*

The "Fundamental Theorem"

Theorem

1. $\mathrm{Trop}(X)$ *is the set of* $w \in \mathbb{R}^n$ *such that* $\mathrm{in}_w X$ *is not empty.*
2. *It is the underlying set of a connected polyhedral complex of pure dimension* $\dim X$.
3. *This polyhedral complex can be chosen such that all faces have rational slope.*

- Versions of this theorem proved by Bergman (1971), Bieri and Groves (1984), Kapranov (2000), Sturmfels (2002), Speyer (2005), and Draisma (2008).

The "Fundamental Theorem"

Theorem

1. $\operatorname{Trop}(X)$ *is the set of* $w \in \mathbb{R}^n$ *such that* $\operatorname{in}_w X$ *is not empty.*
2. *It is the underlying set of a connected polyhedral complex of pure dimension* $\dim X$.
3. *This polyhedral complex can be chosen such that all faces have rational slope.*

- Versions of this theorem proved by Bergman (1971), Bieri and Groves (1984), Kapranov (2000), Sturmfels (2002), Speyer (2005), and Draisma (2008).
- Also follows from elimination of quantifiers for algebraically closed valued fields, proved by Robinson (1956).

A brief history. 1600–1980.

What is tropical geometry, and where does it come from?

Newton polyhedra.

- Newton polygons; Newton's method for solving polynomials. Newton (late 1600s).
- Newton's method over p-adic fields, and other valued fields. Hensel (1904), Dumas (1906), Ostrowski (1935).
- Nondegeneracy with respect to Newton polyhedra, and applications to sparse intersections, hypersurfaces, singularities. Arnold (1975), Bernstein (1975), Kouchnirenko (1975), Khovanskii (1976), Varchenko (1976).

A brief history. 1970-2000.

What is tropical geometry, and where does it come from?

Limits of logarithms, idempotent algebras, and computational applications.

- Logarithmic limit sets of algebraic varieties. Bergman (1971), Bieri and Groves (1984).
- "Tropical semirings" and applications to optimization and computer science. Simon (1988).
- Amoebas of real and complex varieties. Gelfand, Kapranov, and Zelevinsky (1994).
- Idempotent analysis and dequantization. Litvinov and Maslov (1998).
- Dequantization of algebraic geometry on logarithmic paper. Viro (2000).

A brief history. Early 2000s.

Tropical geometry emerges as a field of research.

- Nonarchimedean amoebas. Kapranov (2000), Kontsevich (2000).
- Connections to Gröbner theory. Sturmfels (2002).
- Complex amoebas and Monge-Ampère measures. Passare and Rullgård (2004).
- Complex enumerative geometry of curves. Mikhalkin (2005).
- Real enumerative geometry of curves. Itenberg, Kharlamov, and Shustin (2003).
- The tropical Grassmannian, and Bergman fans of matroids. Speyer and Sturmfels (2004), Ardila and Klivans (2006)

A brief history. Late 2000s

More recent developments.

- Tropical proofs of WDVV and Caporaso-Harris formulas. Gathmann and Markwig (2007, 2008).
- Tropical Riemann-Roch, Abel-Jacobi, and Torelli theorems. Baker and Norine (2007), Gathmann and Kerber (2008), Mikhalkin and Zharkov (2008), Caporaso and Viviani (2010), Branetti, Melo, and Viviani (2011).
- New results in enumerative geometry of curves. Brugallé and Mikhalkin (2007), Fomin and Mikhalkin (2010).
- Discriminants, implicitization, and computational algebraic geometry. Bogart, Jensen, Speyer, Sturmfels, and Thomas (2007), Dickenstein, Feichtner, and Sturmfels (2007), Sturmfels, Tevelev, and Yu (2007), Cueto (2011).

A brief history. Late 2000s

Connections with other fields of mathematics.

- Algebraic dynamics. Einsiedler, Kapranov, and Lind (2006).
- Number theory, Bogomolov's conjecture, and canonical subgroups. Gubler (2007), Rabinoff (2010).
- Birational geometry and minimal model program. Tevelev (2005), Hacking, Keel, and Tevelev (2006, 2009).
- Brill-Noether theory. Baker (2008), Cools, Draisma, P, and Robeva (2010), Caporaso (2011).
- Hodge structures and weight filtrations. Hacking (2008), Helm and Katz (2008), P (2009), Katz and Stapledon (2010).
- Mirror symmetry. Gross, Siebert, and Pandharipande (2010).

A maximally complete field

Fix $K = k((t^{\mathbb{R}}))$ field of transfinite series, with k algebraically closed. Elements of K are formal sums

$$a = \sum_{i \in \mathbb{R}} a_i t^i$$

with well-ordered support.

The field K is

- algebraically closed.
- complete with respect to $\nu(a) = \min\{i \in \mathbb{R} \mid a_i \neq 0\}$.
- spherically complete, with value group $\mathbb{R}$.

Analytification

Let X be an affine variety over K, and set $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$.

Definition

The analytification X_{an} is the set of valuations on $K[X]$ that extend ν, with the topology induced from $\mathbb{R}_\infty^{K[X]}$.

Analytification

Let X be an affine variety over K, and set $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$.

Definition

The analytification X_{an} is the set of valuations on $K[X]$ that extend ν, with the topology induced from $\mathbb{R}_\infty^{K[X]}$.

Theorem (Berkovich 1990, Hrushovski and Loeser 2010)

1. *X_{an} is Hausdorff, path connected, locally compact, and locally contractible.*
2. *X_{an} admits a strong deformation retract onto a finite simplicial complex.*
3. *X_{an} contains $X(K)$ as a dense, totally disconnected subset.*

Example: Analytification of the affine line

Points of the analytic affine line.

Type I. Each point $x \in \mathbb{A}^1(K)$, gives a valuation η_x:

$$\eta_x(f) = \nu(f(x)).$$

Example: Analytification of the affine line

Points of the analytic affine line.

Type I. Each point $x \in \mathbb{A}^1(K)$, gives a valuation η_x:

$$\eta_x(f) = \nu(f(x)).$$

Type II. Each disc of radius r

$$U_r(x) = \{y \in \mathbb{A}^1(K) \mid \nu(y - x) \geq r\}.$$

gives a valuation:

$$\eta_{x,r}(f) = \min\{\nu(f(y) \mid y \in U_r(x)\}.$$

Example: Analytification of the affine line

Points of the analytic affine line.

Type I. Each point $x \in \mathbb{A}^1(K)$, gives a valuation η_x:

$$\eta_x(f) = \nu(f(x)).$$

Type II. Each disc of radius r

$$U_r(x) = \{y \in \mathbb{A}^1(K) \mid \nu(y - x) \geq r\}.$$

gives a valuation:

$$\eta_{x,r}(f) = \min\{\nu(f(y) \mid y \in U_r(x)\}.$$

Proposition (Berkovich 1990)

These are all of the points in $\mathbb{A}^1_{an}$.

Metric on the analytic affine line

Proposition

There is a unique metric on $\mathbb{A}^1_{an} \smallsetminus \mathbb{A}^1(K)$ such that

$$d(\eta_{x,r}, \eta_{x,R}) = |R - r|.$$

With this metric, $\mathbb{A}^1_{an} \smallsetminus \mathbb{A}^1(K)$ is a $\mathbb{R}$-tree.

Caution: The metric topology on $\mathbb{A}^1_{an} \smallsetminus \mathbb{A}^1(K)$ is much finer than the subspace topology.

Tropicalization of analytic spaces

Suppose X is a subvariety of the torus T.

Proposition

The tropicalization map $\mathrm{Trop} : X(K) \to \mathbb{R}^n$ extends to a proper, continuous, surjective map

$$X_{an} \to \mathrm{Trop}(X).$$

The map is given by $\mathrm{Trop}(\eta) = (\eta(x_1), \ldots, \eta(x_n))$.

Tropicalization of analytic spaces

Suppose X is a subvariety of the torus T.

Proposition

The tropicalization map $\mathrm{Trop} : X(K) \to \mathbb{R}^n$ extends to a proper, continuous, surjective map

$$X_{an} \to \mathrm{Trop}(X).$$

The map is given by $\mathrm{Trop}(\eta) = (\eta(x_1), \ldots, \eta(x_n))$.

- The fibers of Trop are "affinoid analytic domains" of dimension equal to $\dim(X)$.
- The fiber over w has a formal model with special fiber $\mathrm{in}_w X$.

Global geometry

Tropicalization and analytification constructions extend naturally to varieties over K that are not necessarily affine.

- Global analytifications of varieties over K, given by gluing analytifications of affine open subsets.
- Global tropicalizations of subvarieties of toric varieties, stratified by tropicalizations of intersections with torus orbits.

Tropicalization of subvarieties of toric varieties

Let Δ be a fan in $\mathbb{R}^n$, with $Y(\Delta)$ the associated toric variety with dense torus T.

The associated tropical space $\mathbb{R}^n(\Delta)$ is a partial compactification of $\mathbb{R}^n$, stratified by linear spaces.

Tropicalization of subvarieties of toric varieties

Let Δ be a fan in $\mathbb{R}^n$, with $Y(\Delta)$ the associated toric variety with dense torus T.

The associated tropical space $\mathbb{R}^n(\Delta)$ is a partial compactification of $\mathbb{R}^n$, stratified by linear spaces.

For each subvariety $X \subset Y(\Delta)$ there is a canonical proper, continuous map $\text{Trop} : X_{an} \to \mathbb{R}^n(\Delta)$.

- The image $\text{Trop}(X)$ meets each linear stratum in a finite polyhedral complex.
- If X meets the dense torus T, then $\text{Trop}(X)$ is the closure of $\text{Trop}(X \cap T)$.

Metrics on tropicalizations of curves

If X is a curve in $Y(\Delta)$, then $\mathrm{Trop}(X \cap T)$ is a connected union of finitely many segments and rays with rational slopes.

- The length of an embedded segment is the **lattice length**.
- The global metric on $\mathrm{Trop}(X \cap T)$ is the shortest path metric.
- The tropicalization map from $X_{an} \smallsetminus X(K)$ to $\mathrm{Trop}(X)$ is surjective, but very far from being an isometry (even on subsets where it restricts to a homeomorphism).

Functoriality Lemma

Fix a variety X over K. Let

$$\iota : X \hookrightarrow Y(\Delta) \quad \text{and} \quad \iota' : X \hookrightarrow Y(\Delta')$$

be closed embeddings into toric varieties, and let

$$\varphi : Y(\Delta) \to Y(\Delta')$$

be an equivariant morphism of toric varieties.

Lemma

If $\varphi \circ \iota = \iota'$ then the induced "linear" function

$$\mathrm{Trop}(\varphi) : \mathbb{R}^n(\Delta) \to \mathbb{R}^{n'}(\Delta')$$

maps $\mathrm{Trop}(\iota(X))$ *surjectively onto* $\mathrm{Trop}(\iota(X'))$.

Limits of tropicalizations

Theorem (P 2009)

The induced map

$$X_{an} \to \varprojlim_{\iota} \mathrm{Trop}(\iota(X))$$

is a homeomorphism.

Limits of tropicalizations

Theorem (P 2009)

The induced map

$$X_{an} \to \varprojlim_{\iota} \mathrm{Trop}(\iota(X))$$

is a homeomorphism.

Theorem (Baker, P, and Rabinoff 2011)

If X is a curve, then the induced map

$$X_{an} \smallsetminus X(K) \to \varprojlim_{\iota} \mathrm{Trop}(\iota(X) \cap T_{\iota})$$

is an isometry.

Semistable decomposition

Let X be a curve over K. Then X_{an} can be decomposed into pieces that locally look like pieces of the affine line.

- Open disc $D = \{\eta \in \mathbb{A}^1_{an} \mid \eta(t) > 0\}$.
- Open annulus $A(R) = \{\eta \in \mathbb{A}^1_{an} \mid R > \eta(t) > 0\}$.

Theorem (Bosch and Lütkebohmert (1985), Temkin (2010))

There is a finite subset $V \subset X_{an} \setminus X(K)$ such that

$$X_{an} \setminus V \;\cong\; \bigsqcup_{fin.} A(R_i) \;\; \bigsqcup_{inf.} D$$

Metrics on analytic curves

Let X be a curve over K, and let $V \subset X_{an} \setminus X(K)$ be a semistable decomposition set.

The metrics on D and $A(R_i)$ induce a shortest path metric on $X_{an} \setminus X(K)$.

Proposition

The induced metric on $X_{an} \setminus X(K)$ is independent of all choices.

Skeletons

The annulus $A(R)$ deformation retracts onto the open segment of length R

$$\Sigma_R = \{\eta_{x,r} \mid R > r > 0\}.$$

Definition

The skeleton $\Sigma(V)$ is the union of V and the open segments Σ_{R_i}.

Proposition

The skeleton $\Sigma(V)$ is a closed embedded metric subgraph in $X_{an} \setminus X(K)$.

A more precise isometry statement

Let X be a curve over K.

Theorem (Baker, P, and Rabinoff 2011)

For any finite embedded subgraph $\Gamma \subset X_{an} \smallsetminus X(K)$ *there exists an embedding* $\iota : X \hookrightarrow Y(\Delta)$ *such that*

1. Trop *maps* Γ *isometrically onto its image.*
2. *For each edge* e *in* $\mathrm{Trop}(\Gamma)$*, the preimage* $\mathrm{Trop}^{-1}(e)$ *is the disjoint union of* e *and an infinite collection of open discs, each of which is contracted by* Trop.

Furthermore, the set of all such embeddings is stable and hence cofinal in the inverse system.

Eso es todo amigos

¡Muchas gracias por su paciencia!