A tropical approach to non-archimedean Arakelov theory

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References

This is a report on joint work with Klaus Künnemann in

- [GK] W. Gubler, K. Künnemann: A tropical approach to non-archimedean Arakelov theory. arXiv:1406.7637

Further references:

- [Gu] W. Gubler: Forms and currents on the analytification of an algebraic variety (after Chambert–Loir and Ducros): arXiv:1303.7364
- [CD] A. Chambert-Loir, A. Ducros: Formes différentielles réelles et courants sur les espaces de Berkovich. arXiv: 1204.6277

Arakelov theory

X regular projective variety over number field K.

- Algebraic intersection theory $\rightsquigarrow$ geometric information about X, e.g. degree.
- Arithmetic intersection theory with arithmetic information about X, e.g. *height*.
  - $\dim X = 1$: Arakelov, Faltings
  - $\dim X \geq 2$: Gillet-Soulé

Arithmetic intersection theory

Idea:

- Assume X has regular projective model $\mathfrak{X}/O_K$.
- $X(\mathbb{C})$ complex manifold.
- g_Z *Green current* for cycle $Z :\overset{on\, X(\mathbb{C})}{\Longleftrightarrow} dd^c g_Z = [\omega_Z] - \delta_Z$ for a differential form ω_Z. Here, $dd^c := \frac{i}{2\pi}\partial\bar{\partial}$ and δ_Z is the current of integration over $Z(\mathbb{C})$.
- $(\mathfrak{Z}, g_Z)$ *arithmetic cycle* $:\Longleftrightarrow \mathfrak{Z}$ cycle on $\mathfrak{X}$ with generic fibre Z, g_Z Green current for Z.
- $f \in K(X)^\times \Rightarrow \widehat{div}(f) := (div(f), -\log|f|)$ is arithmetic cycle.
- *arithmetic intersection product* for $(\mathfrak{Y}, g_Y)$ and $(\mathfrak{Z}, g_Z)$ st Y intersects Z properly in X:

$$(\mathfrak{Z}, g_Z).(\mathfrak{Y}, g_Y) := (\mathfrak{Y}.\mathfrak{Z}, \underbrace{g_Y \wedge \delta_Z + \omega_Y \wedge g_Z}_{=: g_Y * g_Z})$$

Problems

- Need regular models (resolution of singularities is often unknown)
- Canonical heights (e.g. on abelian varieties) cannot be described as an arithmetic intersection number.

Dream

Use analytic spaces over K_v for $v \nmid \infty$ instead of models and a similar analytic theory of currents.

Tropical geometry and Lagerberg's superforms

- *polyhedron* Δ in $\mathbb{R}^r$:
  $\Delta := \bigcap_{i=1}^{N} \{\omega \in \mathbb{R}^r | \langle u_i, \omega \rangle \geq \gamma_i\}, u_i \in \mathbb{Z}^r, \gamma_i \in \mathbb{R}$
- *polyhedral complex* Σ in $\mathbb{R}^r$: finite set Σ of polyhedra in $\mathbb{R}^n$ st
  a) $\Delta \in \Sigma \ \Rightarrow$ every face of Δ is in Σ
  b) $\Delta, \Delta' \in \Sigma \ \Rightarrow \Delta \cap \Delta'$ is face of Δ and Δ'
- Let Σ be of pure dim. n, $\Sigma_n := \{\Delta \in \Sigma | \dim(\Delta) = n\}$.
  A *weight* m on Σ is $m : \Sigma_n \to \mathbb{Z}$.

Lagerberg's superforms

$A^{p,q}(\mathbb{R}^r) := C^\infty(\mathbb{R}^r) \otimes_{\mathbb{Z}} \bigwedge^p (\mathbb{Z}^r)^* \otimes_{\mathbb{Z}} \bigwedge^q (\mathbb{Z}^r)^*$
$\rightsquigarrow$ bigraded differential alternating algebra wrt d', d'':

In coordinates and with multiindex notation

$$\begin{aligned}
\alpha &= \sum_{|I|=p,|J|=q} f_{IJ} d'x_I \wedge d''x_J \\
d'\alpha &= \sum_{i=1}^{r} \sum_{I,J} \frac{\partial f_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J \\
d''\alpha &= \sum_{j=1}^{r} \sum_{I,J} \frac{\partial f_{IJ}}{\partial x_j} d''x_j \wedge d'x_I \wedge d''x_J.
\end{aligned}$$

Tropical geometry and Lagerberg's superforms

- Integration of $\alpha \in A_c^{n,n}(\mathbb{R}^r)$ over n-dim polyhedron Δ is well defined
  $\rightsquigarrow$ current $\delta_\Delta \in D_{n,n}(\mathbb{R}^r) =: D^{r-n,r-n}(\mathbb{R}^r)$
- For a weighted polyhedral complex (Σ, m) of pure dimension n
  $\rightsquigarrow$ current $\delta_{(\Sigma,m)} = \sum_{\Delta \in \Sigma_n} m_\Delta \delta_\Delta \in D_{n,n}(\mathbb{R}^r)$
- (Σ, m) is a *tropical cycle* $\Leftrightarrow d'\delta_{(\Sigma,m)} = 0$ ($\Leftrightarrow$ balancing condition)

Facts from tropical geometry:

From now on, K is an algebraically closed field which is complete with respect to a non-trivial non-archimedean absolute value.

- There is a well-defined intersection product of tropical cycles on $\mathbb{R}^r$ (no equivalence needed)
- For closed subvariety U of $T = \mathbb{G}_m^r$ over K, let $\mathrm{trop} : T^{\mathrm{an}} \to \mathbb{R}^r, t \mapsto (v(t_1), \ldots, v(t_r))$ and $\mathrm{Trop}(U) := \mathrm{trop}(U^{\mathrm{an}})$.

Then $\mathrm{Trop}(U)$ is a *tropical cycle* with canonical weights.

δ-preforms

Definition

A current in $D^{p,q}(\mathbb{R}^r)$ is a δ*-preform* of type (p,q)
$:\Leftrightarrow$ it is of the form

$$\sum_{i=1}^{N} \alpha_i \wedge \delta_{C_i}$$

with $\alpha_i \in A^{p_i,q_i}(\mathbb{R}^r)$ and C_i tropical cycle of codimension k_i with $(p,q) = (p_i + k_i, q_i + k_i)$
$\rightsquigarrow$ bigraded differential algebra $P^{\bullet,\bullet}(\mathbb{R}^r)$ wrt d', d'', where $\delta_{C_i} \wedge \delta_{C'_j} := \delta_{C_i \cdot C'_j}$ using the tropical intersection product.

Tropical charts

X algebraic variety over K.

Definition

A *tropical chart* (V, φ_U) consists of

- open subset U of X
- closed immersion $\varphi_U : U \hookrightarrow T := \mathbb{G}_m^r$,
  $\mathrm{trop}_U := \mathrm{trop} \circ \varphi_U : U^{\mathrm{an}} \to N_U := \mathbb{R}^r$
- open subset Ω of $\mathrm{Trop}(U)$ st. $V := \mathrm{trop}_U^{-1}(\Omega)$ (open in U^{an})

Tropical charts form a basis for X^{an}.

δ-forms and δ-currents

Definition

A *δ-form* α on X^{an} is given by tropical charts $(V_i, \varphi_{U_i})_{i \in I}$ covering X^{an} and $\alpha_i \in P^{\bullet,\bullet}(N_{U_i})$ st

$$\alpha = \alpha' \Leftrightarrow \alpha_i\Big|_{V_i \cap V_j'} = \alpha_j'\Big|_{V_i \cap V_j'} \quad \text{in a tropical sense (see [GK])}$$

$\rightsquigarrow$ bigraded differential alternating algebra $B^{\bullet,\bullet}(X^{\mathrm{an}})$ wrt d', d''.

Topological dual $B_C^{\bullet,\bullet}(X^{\mathrm{an}})$ is $E^{\bullet,\bullet}(X^{\mathrm{an}})$ =: space of *δ-currents*.

Using only Lagerberg's superforms and skipping tropical cycles $\overset{\textit{similarly}}{\rightsquigarrow}$ smooth (p, q)-forms of Chambert-Loir and Ducros (see [CD], [Gu]) leading to a subalgebra $A^{\bullet,\bullet}(X^{\mathrm{an}})$ of $B^{\bullet,\bullet}(X^{\mathrm{an}})$.

δ-forms are analogous to complex differential forms with logarithmic singularities.

First Chern δ-current form

L line bundle in X, $\| \quad \|$ continuous metric in L^{an}.
For a local frame of t of L on open subset U of X,

$$[c_1(L, \| \quad \|)] := d'd''[-\log \|t\|]$$

is a δ-current on U^{an} independent of the choice t.

By a partition of unity argument, we get a well-defined δ-current $[c_1(L, \| \quad \|)]$ called the *first Chern δ-current*.

Similarly as in [CD], we have

Poincaré-Lelong formula

s meromorphic section of L

$$\Rightarrow d'd''[-\log \|s\|] = [c_1(L, \| \quad \|)] - \delta_{div(s)} \quad \text{as } \delta-\text{currents.}$$

Model-metrics and δ-metrics

Important are *model-metrics*: (X, L) generic fibre of $(\mathfrak{X}, \mathcal{L})$
$x \in X^{\mathrm{an}} \rightsquigarrow \mathrm{red}(x) \in \mathfrak{X}_s$ (special fiber)
Choose trivialization $\mathcal{L}\big|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$ at $\mathrm{red}(x)$:
$s \in \Gamma(\mathcal{U}, \mathcal{L}) \leftrightarrow \gamma \in \mathcal{O}(\mathcal{U})$
model metric $\| \ \|_{\mathcal{L}}$ is given at x by $\|s(x)\|_{\mathcal{L}} := |\gamma(x)|$.

Definition

- Metric $\| \ \|$ on L is called *smooth* $:\Leftrightarrow -\log \|t\| \in A^{0,0}(U)$ for any local trivialization $t : U \to L$.
- A δ*-metric* on L is a continuous metric $\| \ \|$ st the δ-current $[c_1(L, \| \ \|)]$ is represented by a δ-form $c_1(L, \| \ \|)$.

Fact: A model metric is a δ-metric, but it is not always smooth!

Chambert-Loir measures

- Chambert–Loir introduced a discrete measure on X^{an} related to the model metric $\| \quad \|_{\mathcal{L}}$.
- Defined by using degrees of the irreducible components of $\mathfrak{X}_s$.
- Important to describe equidistribution measures in arithmetic geometry.

Theorem (GK)

If X is proper variety K of dimension n, then $c_1(L, \| \quad \|_{\mathcal{L}})^{\wedge n}$ is a Radon measure on X^{an} equal to the Chambert-Loir measure from arithmetic geometry.

Remark: In [CD], there is a similar theorem. Note however, that they need an approximation process by smooth metrics to make sense of the wedge products of the first Chern currents.

Non-archimedean Arakelov theory

X proper variety over K of dimension n.

Definition

A δ-current g_Z is called a Green current for cycle Z
$:\Leftrightarrow \ d'd''g_Z = [\omega_Z] - \delta_Z$

$\overset{\text{Poincaré}}{\underset{\text{LeLong}}{\rightsquigarrow}}$ $g_D := -[\log \|s\|]$ is a Green current for $D := div(s)$ and δ-metric $\| \quad \|$.

Proposition [GK]

If D intersects Z property, then

$$g_D * g_Z := g_D \wedge \delta_Z - c_1(L, \| \quad \|) \wedge g_Z$$

is a Green current for $D.Z$.

Proof follows from Poincaré-Lelong.

Local heights

Definition

Let $D_0, \dots, D_n$ be Cartier divisors intersecting properly on X, then

$$\lambda(X) := \langle g_{D_0} * \cdots * g_{D_n}, 1 \rangle$$

is called the *local height* of X (using δ-metrics on $\mathcal{O}(D_0), \dots, \mathcal{O}(D_n)$).

Theorem (GK)

If we use model-metrics on $\mathcal{O}(D_0), \dots, \mathcal{O}(D_n)$, then $\lambda(X)$ is the usual local height of X in arithmetic geometry given as the intersection number of the Cartier divisors on a corresponding model.

Proof of Theorem 2

Proof: Obvious for $n = 0$. Induction formula shows that $\lambda(X)$ is

$$\lambda(D_n) - \int_{X^{\mathrm{an}}} \log \|s_{D_n}\| c_1(\mathcal{O}(D_0), \|\ \ \|_0) \wedge \cdots \wedge c_1(\mathcal{O}(D_n), \|\ \ \|_n)$$

which holds for both variants of local heights using Theorem 1.
□

Dream becomes true for divisors!