HOPF ALGEBRAS IN THE COHOMOLOGY OF $\mathcal{A}_g$, $\mathrm{GL}_n(\mathbb{Z})$, AND $\mathrm{SL}_n(\mathbb{Z})$

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Abstract. We describe a bigraded cocommutative Hopf algebra structure on the weight zero compactly supported rational cohomology of the moduli space of principally polarized abelian varieties. By relating the primitives for the coproduct to graph cohomology, we deduce that $\dim H_c^{2g+k}(\mathcal{A}_g)$ grows at least exponentially with g for $k = 0$ and for all but finitely many positive integers k. Our proof relies on a new result of independent interest; we use a filtered variant of the Waldhausen construction to show that Quillen's spectral sequence abutting to the cohomology of $BK(\mathbb{Z})$ is a spectral sequence of Hopf algebras. From the same construction, we also deduce that $\dim H^{\binom{n}{2}-n-k}(\mathrm{SL}_n(\mathbb{Z}))$ grows at least exponentially with n, for $k = -1$ and for all but finitely many non-negative integers k.

Contents

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1. Introduction

In this paper, we introduce and study new algebraic structures in the cohomology of the complex moduli space $\mathcal{A} = \bigsqcup_{g\geq 0} \mathcal{A}_g$ of principally polarized abelian varieties of all dimensions $g \geq 0$. Except where stated otherwise, all cohomology and compactly supported cohomology groups will be taken with coefficients in $\mathbb{Q}$.

The proofs of our results on the cohomology of $\mathcal{A}$ involve multiple technical constructions of independent interest. We produce a filtered coproduct on the Waldhausen construction of $BK(\mathbb{Z})$, the de-looping of the K-theory space of $\mathbb{Z}$. We also produce a filtered coproduct on a cubical space of graphs and a filtered map to $BK(\mathbb{Z})$ that respects the relevant structures, inducing a morphism of spectral sequences of Hopf algebras. From the E^1 pages, we obtain new relations between the homology of the commutative graph complex GC_2 and $K(\mathbb{Z})$.

The terms on the E^1 page of Quillen's spectral sequence, induced by the rank filtration on $BK(\mathbb{Z})$, have natural interpretations in terms of the cohomology of $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$. Thus, the same constructions give new structures on the unstable cohomology of these groups.

1.1. A Hopf algebra structure on weight zero cohomology.

The compactly supported cohomology $H^*_c(\mathcal{A})$ is isomorphic to $\bigoplus_g H^k_c(\mathcal{A}_g)$, which we consider as a bigraded $\mathbb{Q}$-vector space, in which $H^k_c(\mathcal{A}_g)$ has bidegree $(g, k-g)$. It inherits a natural mixed Hodge structure from those on $H^*_c(\mathcal{A}_g)$, and the new structures we study are in its weight 0 subspace.

Theorem 1.1. *There is a bigraded Hopf algebra structure on the weight zero subspace $W_0H^*_c(\mathcal{A})$.*

The coproduct in this Hopf structure is commutative, and is induced by the proper maps $\mathcal{A}_g \times \mathcal{A}_{g'} \to \mathcal{A}_{g+g'}$ via pullback. The product we construct is more subtle; in particular, it is not commutative.

We now identify a bigraded subspace of the primitives for the coproduct that is closely related to invariant differential forms on symmetric spaces. Let Ω^*_c denote the vector space over $\mathbb{Q}$ spanned by non-trivial exterior products of symbols ω^{4k+1} for $k \geq 1$. It is bigraded by *genus* and *degree minus genus*, where $\omega^{4k_1+1} \wedge \cdots \wedge \omega^{4k_r+1}$, with $k_1 < \cdots < k_r$, has genus $2k_r + 1$ and degree $(4k_1 + 1) + \cdots + (4k_r + 1)$. Let $\Omega_c^*[-1]$ denote the shift of Ω_c^* in which $\omega^{4k_1+1} \wedge \cdots \wedge \omega^{4k_r+1}$ has genus $2k_r + 1$ and degree $(4k_1 + 1) + \cdots + (4k_r + 1) + 1$. Let $\mathrm{Prim}(H)$ denote the subspace of primitives in a Hopf algebra H.

Theorem 1.2. *There is an injection of bigraded vector spaces*

$$\Omega_c^*[-1] \otimes \mathbb{R} \to \mathrm{Prim}(W_0 H_c^*(\mathcal{A}; \mathbb{R})).$$

This injection uses the identification $W_0 H_c^*(\mathcal{A}_g) \cong H_c^*(A_g^{\mathrm{trop}})$, where A_g^{trop} is the moduli space of principally polarized tropical abelian varieties [BBC+24, BMV11], the stratification by locally symmetric spaces $A_g^{\mathrm{trop}} = \bigsqcup_{g' \leq g} P_{g'}/\mathrm{GL}_{g'}(\mathbb{Z})$, and the inclusion of the genus g subspace of $\Omega_c^*[-1]$ in the compactly supported cohomology of $P_g/\mathrm{GL}_g(\mathbb{Z})$, as in [Bro23]. Here P_g denotes the cone of positive definite symmetric bilinear forms on $\mathbb{R}^g$.

The subspace of primitives in a graded Hopf algebra is a Lie algebra with respect to the bracket $[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx$. By relating this Lie algebra to the cohomology of the commutative graph complex GC_2 and the Grothendieck–Teichmüller Lie algebra, we find a subspace of $\Omega_c^*[-1]$, in diagonal bidegree (d, d), that generates a free Lie subalgebra.

Theorem 1.3. *For $N = 11$, the image of $\{\omega^5, \ldots, \omega^{4N+1}\}$ in $\mathrm{Prim}(W_0 H_c^*(\mathcal{A}; \mathbb{R}))$ generates a free Lie subalgebra.*

The statement for $N = 11$ reflects what we can prove with the current knowledge of the Grothendieck–Teichmüller Lie algebra; it is expected to hold for all N.

By the Milnor–Moore theorem, the cocommutative Hopf algebra $W_0 H_c^*(\mathcal{A})$ is isomorphic to $\mathcal{U}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A})))$, the universal enveloping algebra of its Lie algebra of primitives. The universal enveloping algebra on the free Lie algebra generated by the images of $\omega^5, \ldots, \omega^{4N+1}$ is isomorphic to a tensor algebra generated by those elements. Therefore Theorems 1.2 and 1.3 have the following implications for dimensions. Let $\mathcal{T}_*$ denote a free genus-graded tensor algebra with one generator in each genus $3, 5, \ldots, 23$.

Corollary 1.4. *The dimension of $W_0 H_c^{2g}(\mathcal{A}_g)$ is greater than or equal to the dimension of $\mathcal{T}_g$. More generally, whenever $\Omega_c^*[-1]$ is nonzero in bidegree $(g_0, g_0 + k)$ for some nonnegative integer k, the dimension of $H_c^{2g+k}(\mathcal{A}_g)$ is greater than or equal to the dimension of $\mathcal{T}_{g-g_0}$.*

Corollary 1.5. *The dimension of $W_0 H_c^{2g+k}(\mathcal{A}_g)$ grows at least exponentially with g for $k = 0$ and all but finitely many positive integers k.*

There are at most 20 exceptional values of k for which $\dim H_c^{2g+k}(\mathcal{A}_g)$ does not grow at least exponentially; see Section 7.3.1. The precise dimension bounds given by Corollary 1.4 depend on knowing that the image of $\{\omega^5, \ldots, \omega^{4N+1}\}$ generates a free Lie subalgebra for $N = 11$. However, the exponential growth stated in Corollary 1.5 already follows from the statement for $N = 2$. Extending Theorem 1.3 to larger values of N improves the base of the exponential lower bound only slightly; see Section 7.2.

The complex moduli space $\mathcal{A}_g$ is a smooth Deligne–Mumford stack and has the homotopy type of a classifying space for $\mathrm{Sp}_{2g}(\mathbb{Z})$. Thus, each of the results stated above has equivalent reformulations in terms of the group cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Indeed, applying Poincaré duality and identifying the singular cohomology of $\mathcal{A}_g$ with the group cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ gives

$$H^*_c(\mathcal{A}_g) \cong H^{g^2+g-*}(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})^\vee.$$

In particular, we have the following immediate consequence of Corollary 1.5.

Corollary 1.6. *For $k = 0$ and all but finitely many positive integers k, the dimension of $H^{g^2-g-k}(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})$ grows at least exponentially with g.*

It was previously known that $H^{g^2-g}(\mathcal{A}_g)$ is nonvanishing for all g, because the tautological subring of $H^*(\mathcal{A}_g)$, i.e., the subring generated by the λ-classes, is Gorenstein with socle in this degree [vdG99]. It is expected that $H^k(\mathcal{A}_g)$ may vanish for $k > g^2 - g$ [BPS23, Question 1.1]. We note that all classes in the tautological subring are of pure weight, i.e., weight equal to cohomological degree, and the tautological subspace of $H^{g^2-g}(\mathcal{A}_g)$ always has rank 1. The growth we find is in the top weight cohomology, i.e., the graded piece of the weight filtration dual to the weight zero compactly supported cohomology.

1.2. **Filtered coproduct on the Waldhausen construction.** The heart of our proof of Theorems 1.1 and 1.2 is the construction of a coassociative but not cocommutative coproduct on the Waldhausen construction of $BK(\mathbb{Z})$ [Wal85] that is compatible with the rank filtration. Here, $BK(\mathbb{Z})$ is the 1-fold delooping of the K-theory space for $\mathbb{Z}$, so $\pi_{i+1}(BK(\mathbb{Z})) = K_i(\mathbb{Z})$. The homological spectral sequence associated to the rank filtration on $BK(\mathbb{Z})$ is called the *Quillen spectral sequence* and we denote it by ${}^QE^*$. The terms on the E^1 page satisfy

$$ {}^QE^1_{n,k} \cong H_k(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}_n \otimes \mathbb{Q}), \tag{1}$$

where St_n denotes the Steinberg module. Our filtered coproduct gives rise to a bigraded Hopf algebra structure on each page of ${}^QE^*$. Let $\mathrm{Indec}(H)$ denote the indecomposables in a Hopf algebra H. For ${}^QE^1$ we have the following.

Theorem 1.7. *There is a bigraded commutative Hopf algebra structure on the E^1 page of the Quillen spectral sequence*

$${}^QE^1_{n,k} = H_k(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}_n \otimes \mathbb{Q}),$$

and a surjection of bigraded vector spaces

$$\mathrm{Indec}({}^QE^1 \otimes \mathbb{R}) \xrightarrow{\pi} \Omega^*_c[-1]^\vee \oplus \mathbb{R} \cdot e.$$

Here, e is in bidegree $(1, 0)$. Moreover, the graded linear dual of π restricted to $\langle \omega^5, \ldots, \omega^{45} \rangle$ induces an injection from the corresponding free tensor algebra $T(\langle \omega^5, \ldots, \omega^{45} \rangle)$.

1.3. **Spectral sequence of graphical Hopf algebras.** Our proof of Theorem 1.3 relies on relations between the filtered coproduct on the Waldhausen construction of $BK(\mathbb{Z})$ and an analogous filtered coproduct on a cubical space of graphs. This filtered space of graphs has the rational homology of a point, but the graded dual of the E^1 page of the associated spectral sequence is the cohomology of the commutative graph complex GC_2.

This space of graphs admits a filtered map to the Waldhausen construction of $BK(\mathbb{Z})$ that respects all of the relevant structures. The induced map on the diagonal subspace $E_1^{g,g}$ will be most important. Let ${}^QE_*$ denote the cohomological spectral sequence dual to ${}^QE^*$.

Proposition 1.8. *There is a morphism of graded Lie algebras*

$$\mathrm{Prim}\left(\bigoplus_g {}^Q E_1^{g,g}\right) \to H^0(\mathsf{GC}_2) \tag{2}$$

which, after tensoring with $\mathbb{R}$, sends ω^{4k+1} for all $k \geq 1$ to an element that is non-trivial in the abelianisation of $H^0(\mathsf{GC}_2) \otimes \mathbb{R}$ with respect to its graded Lie algebra structure.

The morphism in the previous proposition is induced by the tropical Torelli map. The class ω^{4k+1} pairs non-trivially with the homology class of the wheel graph $W_{2k+1} \in H_0(\mathsf{GC}_2)$. This pairing is given by an integral which is a non-zero rational multiple of $\zeta(2k+1)$ [BS24]. To prove Theorem 1.3, we use known results on the Grothendieck–Teichmüller Lie algebra, which is isomorphic to $H^0(\mathsf{GC}_2)$ by [Wil15], and the fact that it contains a free Lie algebra with one element in odd degree [Bro12].

We pass from such statements about ${}^Q E_*$ to our theorems on $W_0 H_c^*(\mathcal{A}_g)$, as follows. Using one of the main results of [BBC$^+$24], we show that there is a canonical choice of an element e in ${}^Q E_{1,0}^1$ and an isomorphism of bigraded algebras

$$(W_0 H_c^*(\mathcal{A}))^\vee \otimes_{\mathbb{Q}} \mathbb{Q}[x]/x^2 \xrightarrow{\sim} {}^Q E_{*,*}^1 \tag{3}$$

taking x to e. The element e is a primitive in the Hopf algebra structure, and Theorem 1.1 is proved by taking the quotient by the Hopf ideal generated by this primitive, and passing to the graded dual. See Section 4.

1.4. **Applications to the unstable cohomology of $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$.** We now reinterpret the results and constructions discussed above in terms of the cohomology of $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$. Recall that

$$H_*(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}_n \otimes \mathbb{Q}) \cong H^{\binom{n}{2}-*}(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}_{\mathrm{or}})$$

where $\mathbb{Q}_{\mathrm{or}}$ denotes the $\mathrm{GL}_n(\mathbb{Z})$-module given by orientations on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, the symmetric space of positive definite symmetric bilinear forms on $\mathbb{R}^n$ of determinant 1. To relate to $\mathrm{SL}_n(\mathbb{Z})$, Shapiro's Lemma [Wei94, 6.3.2, p. 171] gives

$$H^*(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong H^*(\mathrm{GL}_n(\mathbb{Z}); \mathrm{Ind}_{\mathrm{SL}_n(\mathbb{Z})}^{\mathrm{GL}_n(\mathbb{Z})} \mathbb{Q})$$

and $\mathrm{Ind}_{\mathrm{SL}_n(\mathbb{Z})}^{\mathrm{GL}_n(\mathbb{Z})} \mathbb{Q} \cong \mathbb{Q} \oplus \widetilde{\mathbb{Q}}$ where $\widetilde{\mathbb{Q}}$ denotes the determinantal representation of $\mathrm{GL}_n(\mathbb{Z})$. Now by [EVGS13, Lemma 7.2], the orientation module $\mathbb{Q}_{\mathrm{or}}$ is isomorphic to $\mathbb{Q}$ if n is odd and to $\widetilde{\mathbb{Q}}$ if n is even. Thus, if n is odd then $\mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$, and

$$H^*(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong H^*(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}) \cong H^*(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}_{\mathrm{or}}).$$

If n is even, then

$$H^*(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong H^*(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}) \oplus H^*(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}_{\mathrm{or}}) \cong H^*(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}) \oplus H^*(\mathrm{GL}_n(\mathbb{Z}); \widetilde{\mathbb{Q}}).$$

In particular, our construction of a Hopf algebra structure on Quillen's spectral sequence, together with the injection in Theorem 1.2, gives the following result and produces a bigraded commutative Hopf algebra structure on a large subspace of $\bigoplus_n H^*(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$.

Corollary 1.9. *The dimensions of* $H^{\binom{n}{2}-n-k}(\mathrm{GL}_n(\mathbb{Z});\mathbb{Q}_{\mathrm{or}})$ *and of* $H^{\binom{n}{2}-n-k}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q})$ *grow at least exponentially with* n *for* $k=-1$ *and for all but finitely many nonnegative integers* k.

There are at most 10 exceptional values of k for which these groups do not grow at least exponentially; see Section 7.3.2.

The cohomology groups $H^i(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q})$ are stable in low degrees, for $i \leq n-2$ [LS19]; the growth in Corollary 1.9 takes place entirely in the unstable range. In the highest degrees, $H^{\binom{n}{2}-n-k}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q})$ is conjectured to vanish for $k < -1$ [CFP14, Conjecture 2]. Hence, Corollary 1.9 says that the cohomology groups grow exponentially in the degree immediately below the threshold at and above which they are expected to vanish, and at almost every fixed distance below the threshold.

The following corollary can be deduced from Theorem 1.7 by duality.

Corollary 1.10. *There is an injection of bigraded vector spaces*

$$\mathrm{Sym}\,(\Omega_c^*[-1] \oplus \mathbb{Q}\cdot\epsilon)\otimes\mathbb{R} \hookrightarrow \bigoplus_n H_{\binom{n+1}{2}-*}(\mathrm{SL}_n(\mathbb{Z});\mathbb{R})$$

Here, ϵ *is in bidegree* $(1,0)$*, and* $H_{\binom{n+1}{2}-k}(\mathrm{SL}_n(\mathbb{Z}))$ *is in bidegree* $(n,k-n)$.

Here, and elsewhere, Sym denotes the free graded-commutative algebra on a graded vector space. It is the tensor product of an exterior algebra on elements of $\Omega_c^*[-1]\oplus\mathbb{R}\epsilon$ of odd degree, and a commutative polynomial algebra on the elements of even degree. A version of this statement (without the class ϵ) was announced by Ronnie Lee [Lee78], but no proof has appeared in the literature. Our results imply that the symmetric algebra on a much larger set of primitive elements embeds into the cohomology of the special linear group: in addition to the classes in $\Omega_c^*[-1]\oplus\mathbb{Q}\cdot\epsilon$, we may also take infinitely many independent commutators in the free Lie algebra on $\{\omega^5,\ldots,\omega^{45}\}$; see Remark 7.3.

The injection in Corollary 1.10 is defined with real coefficients. We also give related constructions with rational coefficients. The wheel graphs W_{2k+1}, for positive integers k, give non-trivial homology classes in the graph homology $H_0(\mathsf{GC}_2)$, i.e., the graded dual of $H^0(\mathsf{GC}_2)$. Moreover, $[W_{2k+1}]$ pairs nontrivially with ω^{4k+1}, by pushing forward from the filtered graphical space discussed in Section 1.3 and identifying $[\omega^{4k+1}]$ with a compactly supported cohomology class on the rank-$(2k+1)$ stratum of $BK(\mathbb{Z})$. By showing that these odd wheel classes are primitive with respect to the coproduct, we deduce the following:

Corollary 1.11. *There is an injective map of commutative bigraded algebras*

$$\mathbb{Q}[W_3,W_5,\ldots,W_{2k+1},\ldots]\otimes\mathbb{Q}[e]/(e^2)\longrightarrow {}^{Q}E^1_{*,*}$$

where the element e *is in bidegree* $(1,0)$.

Remark 1.12. Note that the last two corollaries only give polynomial growth in the cohomology of $\mathcal{A}$ and $\mathrm{SL}_n(\mathbb{Z})$. Our results on exponential growth are much stronger and fundamentally use the fact that our co-commutative Hopf algebras are far from commutative.

Remark 1.13. When this paper was in the final stages of preparation, we learned that Ash, Miller, and Patzt independently discovered a bigraded commutative Hopf algebra structure on $\bigoplus_{k,n} H_k(\mathrm{GL}_n(\mathbb{Z});\mathrm{St}_n\otimes\mathbb{Q})$ and described some indecomposable elements. It is not immediately clear whether the two Hopf algebra structures agree. They also found a related Hopf algebra structure on $\bigoplus_{k,n} H_k(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}_n \otimes \widetilde{\mathbb{Q}})$ [AMP24].

1.5. **Relations to moduli spaces of curves.** Theorems 1.1 and 1.3, and the resulting exponential growth of $\dim W_0 H_c^{2g}(\mathcal{A}_g)$ (Corollary 1.5 for $k = 0$) are analogs of the main results of [CGP21] for the moduli space of curves $\mathcal{M} = \bigsqcup_{g \geq 2} \mathcal{M}_g$. Neither of these collections of results appears to be directly deducible from the other. However, $W_0 H_c^*(\mathcal{A})$ and $W_0 H_c^*(\mathcal{M})$ can be related to each other through a zig-zag, as follows. Recall the Torelli map $\mathcal{M} \to \mathcal{A}$, taking a curve to its Jacobian. The Torelli map is not proper, and hence does not give rise to a natural morphism of mixed Hodge structures between $H_c^*(\mathcal{M})$ and $H_c^*(\mathcal{A})$, but it factors as $\mathcal{M} \to \mathcal{M}^{\mathrm{ct}} \to \mathcal{A}$, through the open inclusion of $\mathcal{M}$ into the moduli space $\mathcal{M}^{\mathrm{ct}}$ of curves of compact type (or the image of $\mathcal{M}^{\mathrm{ct}}$ in $\mathcal{A}$). The Torelli map extends to a proper morphism on $\mathcal{M}^{\mathrm{ct}}$, giving rise to a zig-zag of mixed Hodge structures

$$H_c^*(\mathcal{M}) \to H_c^*(\mathcal{M}^{\mathrm{ct}}) \leftarrow H_c^*(\mathcal{A}).$$

However, we do not yet have sufficient understanding of the compactly supported cohomology of $\mathcal{M}^{\mathrm{ct}}$ (or its image in $\mathcal{A}$) to use such a zig-zag effectively.

Nevertheless, the Grothendieck–Teichmüller Lie algebra, via its interpretation as the zeroth degree cohomology of the commutative graph complex GC_2 plays a common role in the top weight compactly-supported cohomology of both $\mathcal{A}$ and $\mathcal{M}$. Indeed, the main results of [CGP21] identify $W_0 H_c^*(\mathcal{M})$ with $H^*(\mathsf{GC}_2)$ and thereby endow it with the structure of a bigraded Lie algebra. Combining this identification with results from Grothendieck–Teichmüller theory [Bro12, Wil15] shows that $\bigoplus_g W_0 H_c^{2g}(\mathcal{M}_g)$ contains a free Lie subalgebra with one generator σ_g in each odd genus $g \geq 3$ and hence $\dim W_0 H_c^{2g}(\mathcal{M}_g)$ grows at least exponentially with g. In particular the graph cohomology $H^*(\mathsf{GC}_2)$ plays the role of an intermediary between $W_0 H_c^*(\mathcal{M})$ and $W_0 H_c^*(\mathcal{A})$. Indeed, the main construction of [CGP21] is a map from GC_2 to a cellular chain complex that computes $W_0 H_c^*(\mathcal{M})$, which is in fact a quasi-isomorphism. Here, we construct a map from $\mathrm{Prim}\left({}^Q E_1\right)$ to $H^*(\mathsf{GC}_2)$ that vanishes on ${}^Q E_1^{1,0}$ and hence factors as

$$\mathrm{Prim}\left({}^Q E_1\right) \to \mathrm{Prim}\left(W_0 H_c^*(\mathcal{A})\right) \to H^*(\mathsf{GC}_2).$$

This map $\mathrm{Prim}\left(W_0 H_c^*(\mathcal{A})\right) \to H^*(\mathsf{GC}_2)$ is not an isomorphism, but is nevertheless essential to our proof of Theorem 1.3 and its corollaries. It remains possible that this map may restrict to an isomorphism $\mathrm{Prim}\left(\bigoplus_g H_c^{2g}(\mathcal{A})\right) \to H^0(\mathsf{GC}_2)$; see Question 1.16.

Remark 1.14. The analog of Corollary 1.5 for $H_c^*(\mathcal{M})$ is an open problem; it is conjectured but not known that $\dim H_c^{2g+k}(\mathcal{M}_g)$ grows at least exponentially with g for all but finitely many $k \geq 0$ [PW24, Conjecture 1.3]. This is proved for a few dozen values of k. In each known case, the exponential growth is established in one fixed graded piece of the weight filtration; see [PW21, Corollary 1.3] and [PW24, Corollary 1.2].

1.6. **Further results and conjectures.** We now state questions, conjectures, and further results related to injectivity, vanishing, and extensions of Hodge structures involving $W_0 H_c^*(\mathcal{A})$. We also state a generalization of our results related to the Hopf algebra on the Quillen spectral sequence (Theorem 1.7 and Proposition 1.8) for rings of integers in number fields.

1.6.1. *Injectivity.* Theorems 1.2 and 1.3 suggest the following conjecture.

Conjecture 1.15. *The inclusion of $\Omega_c^*[-1] \otimes \mathbb{R}$ into the primitives for the coproduct on $W_0H_c^*(\mathcal{A};\mathbb{R})$ induces an injection $T(\Omega_c^*[-1]) \otimes \mathbb{R} \to W_0H_c^*(\mathcal{A};\mathbb{R})$.*

To support this conjecture, we construct a spectral sequence of bigraded Hopf algebras whose E_1 page is $T(\Omega_c^*[-1])$ and show that the abutment of this spectral sequence is isomorphic, as a bigraded vector space, to the abutment of the Quillen spectral sequence. Note that Conjecture 1.15 implies in particular that the tensor algebra on $\{\omega^{4k+1}\}$ injects into $W_0H_c^*(\mathcal{A};\mathbb{R})$, or equivalently, that the classes ω^{4k+1} generate a free Lie subalgebra of $\mathrm{Prim}(W_0H_c^*(\mathcal{A};\mathbb{R}))$. To prove this weaker statement, it would suffice to show that each class ω^{4k+1} maps into the motivic Lie subalgebra of the Grothendieck–Teichmüller Lie algebra [Bro12].

The map in Conjecture 1.15 is injective for $g \leq 9$ and an isomorphism for $g \leq 7$; the former follows from the methods developed in this paper, the latter from the isomorphism (3) and explicit calculations in [EVGS13]. It is not expected to be an isomorphism in general. Indeed, the dual vector space of $\Omega_c^*[-1]$ in genus g is expected to be spanned by cohomology classes for $\mathrm{GL}_g(\mathbb{Z})$ of non-cuspidal (Eisenstein) type. Using automorphic methods, cuspidal cohomology classes for $\mathrm{GL}_g(\mathbb{Z})$ for $g = 79$ and 105 were recently constructed in [BCG23]. Also, starting from $g = 9$, the Euler characteristic of $T(\Omega_c^*[-1])$ and the isomorphism (3) are not compatible with homological Euler characteristic computations for $\mathrm{GL}_g(\mathbb{Z})$ [Hor05, Theorem 3.3].

The situation potentially changes if we restrict to the diagonal $\bigoplus_g W_0H_c^{2g}(\mathcal{A}_g;\mathbb{R})$, which, by our earlier results, is a graded cocommutative Hopf algebra of particular interest.

Question 1.16. *Is the induced map*

$$T\Big(\bigoplus_{k\geq 1} \omega^{4k+1}\mathbb{R}\Big) \to \bigoplus_g W_0H_c^{2g}(\mathcal{A}_g;\mathbb{R})$$

an isomorphism?

As noted above, the injectivity of this map would follow from Drinfeld's conjecture that the motivic Lie algebra surjects onto the Grothendieck–Teichmüller Lie algebra. Assuming the same conjecture and restricting to the Lie algebra of primitives, the analog of Question 1.16 for the cohomology of the moduli stack of curves has an affirmative answer, since $\bigoplus_g W_0H_c^{2g}(\mathcal{M}_g)$ is isomorphic to $H^0(\mathsf{GC}_2)$.

1.6.2. *Vanishing.* Note that $H_c^i(\mathcal{M}_g)$ vanishes for $i < 2g$ [Har86, CFP12, MSS13]. Passing to weight zero, this implies the vanishing of $H^*(\mathsf{GC}_2)$ in negative degrees; see [Wil15, Theorem 1.1] and [CGP21, Theorem 1.4]. It is expected that $H_c^i(\mathcal{A}_g)$ may also vanish for $i < 2g$ [BPS23, Question 1.1]; this is known for $i < g + \max\{2, g\}$ [BS73, Gun00, BPS23].

The vanishing of $H_c^{2g+1}(\mathcal{M}_g)$ for all g is a compelling open question, closely related to results in deformation theory, such as formality of deformation quantization [PW24, Question 1.4]. The analogous statement for $H_c^*(\mathcal{A})$ is also open.

Question 1.17. *Does $H_c^{2g+1}(\mathcal{A}_g)$ vanish for all g?*

The answer is yes for $g \leq 4$. See [Hai02] and [HT12, Corollary 3].

1.6.3. *Extensions of Tate Hodge structures.* Our study of $W_0H^*_c(\mathcal{A})$ also sheds new light on the full mixed Hodge structure $H^*_c(\mathcal{A})$. In particular, we can now show that this mixed Hodge structure contains a nontrivial extension of Tate Hodge structures in genus 3, answering questions of Hain and Looijenga.

Following Namikawa [Nam80], we shall use the notation $\mathcal{A}_g^{\#}$ for the Satake–Baily–Borel compactification of $\mathcal{A}_g$. Hain observed over twenty years ago that the mixed Hodge structure on $H^6(\mathcal{A}_3)$ is an extension of $\mathbb{Q}(-6)$ by $\mathbb{Q}(-3)$ and stated the expectation that it should be a multiple (possibly trivial) of the extension given by $\zeta(3)$ [Hai02, pp. 473–474]. More recently, Looijenga showed that a Tate twist of a nontrivial multiple of this same extension appears in the stable cohomology of the Satake compactification $H^6(\mathcal{A}_\infty^{\#})$ and asked whether the pullback $H^6(\mathcal{A}_\infty^{\#}) \to H^6(\mathcal{A}_3^{\#})$ is injective [Loo17, p. 1370]. Here, we confirm Hain's expectation and give an affirmative answer to Looijenga's question. Moreover, we show that the extension in $H^6(\mathcal{A}_3)$ is nontrivial.

Theorem 1.18. *The restriction map $H^6(\mathcal{A}_\infty^{\#}) \to H^6(\mathcal{A}_3^{\#})$ is injective. Moreover, the image of this restriction map is equal to the image of $H^6_c(\mathcal{A}_3)$ under push forward for the open inclusion $\mathcal{A}_3 \subset \mathcal{A}_3^{\#}$. In particular, the mixed Hodge structure on $H^6_c(\mathcal{A}_3)$ is the nontrivial extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}$ given by a nonzero rational multiple of $\zeta(3)$.*

By Poincaré duality, we see that $H^6(\mathcal{A}_3)$ is the nontrivial extension of $\mathbb{Q}(-6)$ by $\mathbb{Q}(-3)$ given by a nonzero rational multiple of $\zeta(3)$.

Remark 1.19. The motivic structure of $H^6(\mathcal{A}_3)$ (i.e. its associated mixed Hodge structure and ℓ-adic Galois representations) and the extensions of Tate structures in $H^{2g}(\mathcal{A}_g^{\#})$ have generated sustained interest. See, for instance, [CFvdG20, Section 13] for a discussion of the Siegel and Teichmüller modular forms that occur in $H^6(\mathcal{A}_3)$ and $H^6(\mathcal{M}_3)$ and [vdGL21] for a discussion of the differences that appear when working over fields of positive characteristic.

We predict that Theorem 1.18 generalizes to higher genus, as follows.

Conjecture 1.20. *For odd $g \geq 3$, the mixed Hodge structure on $H^{2g}_c(\mathcal{A}_g)$ has a subquotient isomorphic to the extension of $\mathbb{Q}(-g)$ by $\mathbb{Q}$ given by a nonzero rational multiple of $\zeta(g)$.*

As evidence for Conjecture 1.20, we note that $\mathrm{Gr}^{2g}_W H^{2g}_c(\mathcal{A}_g)$ contains a copy of $\mathbb{Q}(-g)$, because the tautological subring of $H^*(\mathcal{A}_g)$ has its socle in degree $g^2 - g$ [vdG99].

1.7. **Quillen spectral sequences for rings of integers in number fields.** Our proof of the existence of a Hopf algebra structure on the Quillen spectral sequence in Theorem 1.7 holds for much more general rings R. A particular case of interest is when $R = \mathcal{O}_K$ is the ring of integers in a number field K. In this setting, its E^1 page was computed by Quillen and satisfies

$$^{Q}E^1_{n,k}(\mathcal{O}_K) \cong \bigoplus_{\mathfrak{a}} H_k(\mathrm{GL}(P_{\mathfrak{a}}); \mathrm{St}(P_{\mathfrak{a}} \otimes K))$$

where the sum is over representatives $P_{\mathfrak{a}}$ of the isomorphism classes of projective $\mathcal{O}_K$-modules of rank n (which are in bijection with the ideal class group of $\mathcal{O}_K$). Our results imply that ${}^{Q}E(\mathcal{O}_K)$, and in particular its E^1 page, has the structure of a commutative bigraded Hopf algebra. It gives a spectral sequence of Hopf algebras converging to the rational homology of $BK(\mathcal{O}_K)$, which was computed by Borel.

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2. Preliminaries

2.1. **Weight zero cohomology and the tropical moduli space.** Let $\mathcal{A}_g$ denote the complex moduli stack of principally polarized abelian varieties of dimension g. For background on geometry and topology of $\mathcal{A}_g$ and its compactifications, we refer to the articles [Gru09] and [HT18]. For each $k \geq 0$, its compactly supported rational cohomology $H^k_c(\mathcal{A}_g;\mathbb{Q})$ admits a *weight filtration*

$$W_0H^k_c(\mathcal{A}_g;\mathbb{Q}) \subset W_1H^k_c(\mathcal{A}_g;\mathbb{Q}) \subset \cdots \subset H^k_c(\mathcal{A}_g;\mathbb{Q})$$

as part of the mixed Hodge structure on $H^k_c(\mathcal{A}_g;\mathbb{Q})$. Let $\mathcal{A} = \coprod_{g\geq 0}\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of any dimension. Then $H^*_c(\mathcal{A}) = \bigoplus_{g\geq 0} H^*_c(\mathcal{A}_g)$ and we take $W_kH^*_c(\mathcal{A}) = \bigoplus_g W_kH^*_c(\mathcal{A}_g)$.

There is no single canonical choice of normal crossings compactification for $\mathcal{A}_g$. Rather, by [AMRT75], there is a toroidal compactification $\overline{\mathcal{A}_g}^{\Sigma}$ for every choice of certain polyhedral data Σ, as we now recall. Let $P_g \subset \mathrm{Sym}^2((\mathbb{R}^g)^\vee)$ denote the set of symmetric bilinear forms on $\mathbb{R}^g$ that are positive definite. It is an open convex cone of full dimension inside the $\binom{g+1}{2}$-dimensional Euclidean vector space $\mathrm{Sym}^2((\mathbb{R}^g)^\vee)$. Let P^{rt}_g denote the set of positive semidefinite forms on $\mathbb{R}^g$ whose kernel is rational, i.e., the kernel is of the form $W\otimes\mathbb{R}$ for a vector subspace W of $\mathbb{Q}^g$. Thus $P_g \subset P^{\mathrm{rt}}_g$. Let Σ denote any *admissible decomposition* of P^{rt}_g. That is, Σ is an infinite rational polyhedral cone decomposition supported on P^{rt}_g, whose cones are permuted under the action given by $X\cdot A := A^TXA$ for $A \in \mathrm{GL}_g(\mathbb{Z})$, and there are only finitely many orbits of cones of Σ under this action. It is a nontrivial, but classical, fact that admissible decompositions exist. Famous examples include the decomposition into perfect cones; see [AMRT75] and references therein.

It will be convenient to topologize P^{rt}_g not with its subspace topology induced from the ambient vector space $\mathrm{Sym}^2((\mathbb{R}^g)^\vee)$, but rather with its *Satake* topology, which we now define. Let Σ denote any admissible decomposition of P^{rt}_g. The Satake topology on the set P^{rt}_g is the finest topology such that for every cone σ in Σ, the map $\sigma \to P^{\mathrm{rt}}_g$ is continuous. The

Satake topology agrees with the Euclidean topology on P_g^{rt} for $g = 0, 1$, but is strictly finer for $g \geq 2$. On the other hand, the Satake topology restricts to the Euclidean topology on the open subset P_g, since every point of P_g is contained in only finitely many cones of Σ. Moreover, it is independent of the choice of Σ, since any two choices of admissible decompositions admit a common refinement [FC90, IV.2, p. 97]. Throughout, we consider P_g^{rt} as a topological space with the Satake topology.

Now define
$$A_g^{\mathrm{trop}} := P_g^{\mathrm{rt}}/\mathrm{GL}_g(\mathbb{Z}).$$
For an interpretation of A_g^{trop} as a moduli space of principally polarized tropical abelian varieties, see [BMV11]. A comparison theorem, as in [CGP21, Theorem 5.8], implies the following.

Proposition 2.1. [BBC$^+$24, Theorem 3.1], [OO21, Corollary 2.9] *For each $k \geq 0$, there is a canonical isomorphism*
$$H_c^k(A_g^{\mathrm{trop}}) \cong W_0 H_c^k(\mathcal{A}_g).$$

Lemma 2.2. *For each $g > 0$, we have a map $A_{g-1}^{\mathrm{trop}} \to A_g^{\mathrm{trop}}$ which is a homeomorphism onto the closed subspace of A_g^{trop} whose complement is $P_g/\mathrm{GL}_g(\mathbb{Z})$.*

Proof. Consider the linear injection $\mathrm{Sym}^2((\mathbb{R}^{g-1})^\vee) \to \mathrm{Sym}^2((\mathbb{R}^g)^\vee)$ induced by the map $(\mathbb{R}^{g-1})^\vee \to (\mathbb{R}^g)^\vee$ that extends by zero on the last basis vector in the standard ordered basis of $\mathbb{R}^g$. It restricts to a map
$$P_{g-1}^{\mathrm{rt}} \to P_g^{\mathrm{rt}} \tag{4}$$
which we claim is continuous. Indeed, since the Satake topology identifies P_g^{rt} with the colimit of the cones in any admissible decomposition, it suffices to observe that the image of each perfect cone in P_{g-1}^{rt} is a perfect cone in P_g^{rt}. The map (4) descends to a closed embedding
$$A_{g-1}^{\mathrm{trop}} \hookrightarrow A_g^{\mathrm{trop}};$$
the complement of the image is $P_g/\mathrm{GL}_g(\mathbb{Z})$ [BBC$^+$24, Lemma 4.9 and Proposition 4.11]. □

2.1.1. *The tropical spectral sequence.* Let ${}^TE^*$ denote the spectral sequence on Borel–Moore homology, with rational coefficients, associated to the sequence of spaces
$$\emptyset \subset A_0^{\mathrm{trop}} \subset A_1^{\mathrm{trop}} \subset \cdots.$$
We henceforth call ${}^TE$ the *tropical spectral sequence.* Since $A_s^{\mathrm{trop}} \setminus A_{s-1}^{\mathrm{trop}}$ is homeomorphic to $P_s/\mathrm{GL}_s(\mathbb{Z})$ (Lemma 2.2), we have
$${}^TE^1_{s,t} = H^{\mathrm{BM}}_{s+t}(P_s/\mathrm{GL}_s(\mathbb{Z}); \mathbb{Q}). \tag{5}$$

Remark 2.3. Alternatively, let $A_g^{\mathrm{trop}} \cup \{\infty\}$ denote the one-point compactification of A_g^{trop}. Then ${}^TE^*$ is the first quadrant spectral sequence on reduced rational homology of the sequence of pointed spaces
$$\{\infty\} \subset A_0^{\mathrm{trop}} \cup \{\infty\} \subset A_1^{\mathrm{trop}} \cup \{\infty\} \cup \cdots, \tag{6}$$
with ${}^TE^1_{s,t} = H_{s+t}(A_s^{\mathrm{trop}} \cup \{\infty\}, A_{s-1}^{\mathrm{trop}} \cup \{\infty\}; \mathbb{Q}) \cong \widetilde{H}_{s+t}(P_s/\mathrm{GL}_s(\mathbb{Z}) \cup \{\infty\}; \mathbb{Q})$ for $s \geq 0$. The one-point compactification point of view will play a role in Section 2.5, where the associated

graded spaces of the filtered space $A_\infty^{\mathrm{trop}} \cup \{\infty\}$ arising as the colimit of (6) shall be related to the associated graded spaces of a filtration of the space $BK(\mathbb{Z})$.

By Poincaré duality, ${}^T E^1_{s,t}$ is isomorphic to

$$H^{\binom{s+1}{2}-s-t}(P_s/\mathrm{GL}_s(\mathbb{Z}); \mathbb{Q}_{\mathrm{or}}). \tag{7}$$

Here and throughout, $\mathbb{Z}_{\mathrm{or}}$ is the $\mathrm{GL}_s(\mathbb{Z})$-module induced by the action on orientations of the symmetric space P_s for $\mathrm{GL}_s(\mathbb{R})$, and $\mathbb{Q}_{\mathrm{or}} := \mathbb{Z}_{\mathrm{or}} \otimes \mathbb{Q}$. Note that $\mathbb{Q}_{\mathrm{or}}$ is nontrivial if and only if s is even [EVGS13, Lemma 7.2]. Next, (7) is identified with group cohomology

$$H^{\binom{s+1}{2}-s-t}(\mathrm{GL}_s(\mathbb{Z}), \mathbb{Q}_{\mathrm{or}}).$$

Since $\mathrm{GL}_s(\mathbb{Z})$ is a virtual duality group of virtual cohomological dimension $\binom{s}{2}$, with dualizing module $\mathrm{St}_s \otimes \mathbb{Z}_{\mathrm{or}}$, it follows that ${}^T E^1_{s,t}$ is isomorphic to

$$H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s \otimes \mathbb{Q}). \tag{8}$$

The following proposition is a consequence of acyclicity of inflation [BBC+24, §5], as we now explain.

Proposition 2.4. *The spectral sequence*

$${}^T E^1_{s,t} = H_t(\mathrm{GL}_s(\mathbb{Z}), \mathrm{St}_s \otimes \mathbb{Q}) \tag{9}$$

has exact E^1 page and hence $E^2_{s,t} = 0$ for all s, t.

Proof. Let

$$A_\infty^{\mathrm{trop}} = \cup_{g \geq 0} A_g^{\mathrm{trop}} = \varinjlim_g A_g^{\mathrm{trop}}.$$

Let $C_*^{\mathrm{BM}}(X)$ denote the locally finite chain complex associated to a space X.

Note that ${}^T E$ is the spectral sequence associated to the filtration on $C = \varinjlim_g C_*^{\mathrm{BM}}(A_g^{\mathrm{trop}})$ by

$$F_s C = C_*^{\mathrm{BM}}(A_s^{\mathrm{trop}}).$$

For each $g \geq 0$, let $\widetilde{I}_g \subset P_g^{\mathrm{rt}}$ denote the union of the cones in the $\mathrm{GL}_g(\mathbb{Z})$-orbit of the set

$$\{\sigma + \mathbb{R}_{\geq 0} e_g e_g^T : \sigma \in \Sigma_{g-1}^{\mathrm{perf}}\} \tag{10}$$

Let $I_g = \widetilde{I}_g/\mathrm{GL}_g(\mathbb{Z}) \subset A_g^{\mathrm{trop}}$, henceforth referred to as the *inflation locus.*.

Proposition 2.5. [BBC+24, Theorem 5.15] *The link LI_g has the rational homology of a point.*

Proof. The cellular chain complex for LI_g is the one denoted $I_*^{(g)}$ in op. cit., where it is shown that $I_*^{(g)}$ is acyclic. □

Remark 2.6. In fact LI_g is contractible. This is not in the current literature, but can be proved using a slight modification of Yun's Morse-theoretic enhancement of the proof of acyclicity in [BBC+24, Theorem 5.15]. See [Yun22]. Yun's theorem deals not with LI_g but with the link of a similarly defined *matroidal coloop* locus, consisting of the $\mathrm{GL}_g(\mathbb{Z})$-orbits of the unions of polyhedral cones (10) in which σ is of the form

$$\mathbb{R}_{\geq 0}\langle v_1 v_1^T, \ldots, v_n v_n^T \rangle,$$

where $v_1, \ldots, v_n$ are the column vectors of a $g \times n$ totally unimodular matrix.

Corollary 2.7. $H_*^{\mathrm{BM}}(I_g) = 0$.

Proof. We have $H_*^{\mathrm{BM}}(I_g) = H_*(I_g^+, \infty) = \widetilde{H}_*(S(LI_g)) = \widetilde{H}_{*-1}(LI_g)$, where $S(-)$ denotes suspension. And $\widetilde{H}_{*-1}(LI_g) = 0$ by Proposition 2.5. □

Now the inclusions

$$A_{g-1}^{\mathrm{trop}} \subset I_g \subset A_g^{\mathrm{trop}},$$

are closed and hence proper, and

$$H_k^{\mathrm{BM}}(I_g) = 0 \qquad \text{for all } k \geq 0.$$

It follows that for each k and for each g,

$$H_k^{\mathrm{BM}}(A_{g-1}^{\mathrm{trop}}) \to H_k^{\mathrm{BM}}(A_g^{\mathrm{trop}}) \tag{11}$$

is the zero map, since it factors through $H_k^{\mathrm{BM}}(I_g) = 0$. Now the fact that ${}^T E^2_{*,*} = 0$ follows from the following general proposition. □

Proposition 2.8. *Let (C, d) be a chain complex, and let*

$$0 = F_{-1}C \subset F_0C \subset F_1C \subset \cdots$$

be an increasing filtration on C, such that for each $i, k \geq 0$, the map

$$H_k(F_iC) \to H_k(F_{i+1}C) \tag{12}$$

is zero. Then the spectral sequence of the filtered chain complex

$$E^1_{s,t} = H_{s+t}(F_sC, F_{s-1}C)$$

collapses at $E^2_{s,t} = 0$ for all s and t.

Proof. For each i, the short exact sequence of chain complexes

$$0 \to F_iC \to F_{i+1}C \to F_{i+1}C/F_iC \to 0$$

has associated long exact sequence

$$\begin{aligned} \cdots \quad &\longrightarrow \quad H_k(F_iC) \xrightarrow{0} H_k(F_{i+1}C) \to H_k(F_{i+1}C, F_iC) \\ &\longrightarrow \quad H_{k-1}(F_iC) \xrightarrow{0} H_{k-1}(F_{i+1}C) \to H_{k-1}(F_{i+1}C, F_iC) \to \cdots \end{aligned}$$

which, by (12), splits into short exact sequences

$$0 \to H_k(F_{i+1}C) \to H_k(F_{i+1}C, F_iC) \to H_{k-1}(F_iC) \to 0$$

for each $k \geq 0$. Now the rows of ${}^T E^1$ of the spectral sequence read

$$\cdots \leftarrow H_{s+t-1}(F_{s-1}C, F_{s-2}C) \leftarrow H_{s+t}(F_sC, F_{s-1}C) \leftarrow H_{s+t+1}(F_{s+1}C, F_sC) \leftarrow \cdots$$

and fit into the commuting diagram below,

$$\begin{tikzcd}
& & & & & 0 \arrow[dll] \\
& & & H_{s+t}(F_sC) \arrow[dl, hook] & & \\
H_{s+t-1}(F_{s-1}C, F_{s-2}C) & & H_{s+t}(F_sC, F_{s-1}C) \arrow[ll] \arrow[dl, two heads] & & H_{s+t+1}(F_{s+1}C, F_sC) \arrow[ll] \arrow[ul, two heads] & \\
& H_{s+t-1}(F_{s-1}C) \arrow[ul, hook] \arrow[dl] & & & & \\
0 & & & & &
\end{tikzcd}$$

from which it follows that the rows of ${}^{T}E^1$ are exact. Hence ${}^{T}E^2 = 0$. □

Let $(C^*_{\leq s}, d)$ denote the truncation of a cochain complex (C^*, d), where $C^i_{\leq s} = C^i$ for $i \leq s$ and is zero for $i > s$. The category of cochain complexes over a field k is equivalent to the category of graded $k[x]/(x^2)$-modules, with $\deg(x) = +1$. We record the following standard fact.

Lemma 2.9. *Suppose (C^*, d) is an acyclic cochain complex over a field k. Then*

$$(C^*, d) \cong \Big(\bigoplus_s H_s((C^*_{\leq s}, d))\Big) \otimes k[x]/(x^2)$$

as graded $k[x]/(x^2)$-modules.

It will be convenient to consider the cohomological spectral sequence ${}^{T}E_*$ dual to the tropical spectral sequence ${}^{T}E^*$ in (5). The following is a consequence of Proposition 2.4, by applying Lemma 2.9 to ${}^{T}E_1$.

Theorem 2.10. *We have an isomorphism*

$$({}^{T}E_1, d) \cong \Big(\bigoplus_{s \geq 0} W_0 H_c^{s+t}(\mathcal{A}_s; \mathbb{Q})\Big) \otimes \mathbb{Q}[x]/(x^2)$$

of $\mathbb{Z}^2$-graded cochain complexes with differential in degree $(1, 0)$. Here, $W_0 H_c^{s+t}(\mathcal{A}_s; \mathbb{Q})$ is in bidegree (s, t), and x is in bidegree $(1, 0)$.

Proof. Proposition 2.4 shows that ${}^{T}E_1$ is acyclic. The truncation of ${}^{T}E_1$ after the first s columns is the E_1 page of the spectral sequence that converges on E_2 to $W_0 H_c^{s+t}(\mathcal{A}_s; \mathbb{Q})$, supported in column s. The statement then follows from Lemma 2.9. □

In Proposition 4.4 we shall upgrade Theorem 2.10 to an isomorphism of graded-commutative algebras over $\mathbb{Q}$.

2.2. **The Waldhausen construction and Quillen's spectral sequence.** We briefly review here the Waldhausen $S_\bullet$-construction of K-theory, following the survey [Wei13, IV.8]. The construction applies to arbitrary Waldhausen categories. We shall first recall the definition of a Waldhausen category. We do this only briefly, keeping in mind that our main example shall be the category of finitely generated projective left modules over a ring, as in Example 2.11 below. We refer to [Wei13, II.9] for full and precise definitions.

A *category with cofibrations* is a category $\mathcal{C}$ equipped with a special subcategory of morphisms $\mathrm{co}(\mathcal{C})$ called *cofibrations*. Every isomorphism is a cofibration. There is a distinguished 0 object in $\mathcal{C}$ and the unique map from 0 to any object is a cofibration. Cofibrations are preserved under pushouts along arbitrary morphisms; in particular, such pushouts exist. We denote cofibrations with a feathered arrow $\rightarrowtail$.

A *Waldhausen category* is a category with cofibrations together with a family $w(\mathcal{C})$ of morphisms, called *weak equivalences*. Weak equivalences will be denoted $\xrightarrow{\sim}$. All isomorphisms are weak equivalences. Weak equivalences are closed under composition and satisfy the following gluing condition: for every commutative diagram

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd}$$

the induced map of pushouts

$$B \cup_A C \to B' \cup_{A'} C'$$

is a weak equivalence. The following is our main example of interest.

Example 2.11. Let R be a ring; we do not require that R is commutative. Let Proj_R denote the category of finitely generated projective left R-modules, let $\mathrm{co}(\mathrm{Proj}_R)$ be the injections with projective cokernel, and let $w(\mathrm{Proj}_R)$ be the isomorphisms. Then $(\mathrm{Proj}_R, \mathrm{co}(\mathrm{Proj}_R), w(\mathrm{Proj}_R))$ is a Waldhausen category.

We now review Waldhausen's $S_\bullet$-construction of K-theory, following [Wei13, IV.8]. Let $\mathcal{C}$ be any Waldhausen category. For each $p \geq 0$, define a category $S_p(\mathcal{C})$ whose objects are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r, tail] & P_{0,1} \arrow[r, tail] \arrow[d] & P_{0,2} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & P_{0,p} \arrow[d] \\ & 0 \arrow[r, tail] & P_{1,2} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & P_{1,p} \arrow[d] \\ & & 0 \arrow[r, tail] & \cdots \arrow[r, tail] \arrow[d] & \vdots \arrow[d] \\ & & & 0 \arrow[r, tail] & P_{p-1,p} \arrow[d] \\ & & & & 0 \end{tikzcd} \tag{13}$$

in which the horizontal morphisms are cofibrations and $P_{i,j}$ is a chosen cokernel for the horizontal map $P_i \to P_j$, where $P_i := P_{0,i}$. The arrows in $S_p(\mathcal{C})$ are morphisms $P_\bullet \to Q_\bullet$ of such diagrams that are weak equivalences. By a weak equivalence, we mean a morphism $P_\bullet \to Q_\bullet$ such that each component $P_{i,j} \to Q_{i,j}$ is a weak equivalence.

Remark 2.12. In other sources such as [Wei13, IV.8], what we write as $S_p(\mathcal{C})$ above is written as $wS_p(\mathcal{C})$, while $S_p(\mathcal{C})$ is reserved for the category whose objects are diagrams (13) and whose morphisms are morphisms of diagrams, with no restrictions. This more general notion of morphism in $S_p(\mathcal{C})$ is useful when iterating the $S_\bullet$-construction, which we will not need here. Since the only morphisms in $S_p(\mathcal{C})$ relevant to us are the weak equivalences, we shall write $S_p(\mathcal{C})$ instead of $wS_p(\mathcal{C})$ for the category whose objects are triangular diagrams (13) and whose morphisms are weak equivalences.

Returning to Waldhausen's $S_\bullet$-construction, we define functors $d_i \colon S_\bullet(\mathcal{C}) \to S_{\bullet-1}(\mathcal{C})$ for each i by deleting both the ith row and the ith column in (13) (i.e., the row with objects labeled $P_{i,j}$ and the column with objects labeled $P_{k,i}$) and then re-indexing the remaining terms. Similarly, we define functors $s_i \colon S_\bullet(\mathcal{C}) \to S_{\bullet+1}(\mathcal{C})$ by duplicating both the ith row and the ith column in (13), inserting identity morphisms, and then re-indexing. In this way, $S_\bullet(\mathcal{C})$ is a simplicial object in the category of categories. Taking the nerve yields a bisimplicial set

$$\begin{aligned} N_\bullet S_\bullet(\mathcal{C}) \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} &\to \mathsf{Set} \\ ([p],[q]) &\mapsto N_q S_p(\mathcal{C}). \end{aligned}$$

Remark 2.13. In order to get a bisimplicial *set*, the category $\mathcal{C}$ must be small, i.e. there is a set of objects. In that case $S_\bullet(\mathcal{C})$ is also small. The main examples, such as finitely generated projective left modules over some ring, are *essentially small* but not small if one takes *all* such objects. In those cases one chooses (often tacitly) a small category $\mathcal{C}^{\mathrm{small}}$ equivalent to $\mathcal{C}$ and applies the Waldhausen construction to that. The resulting space $|N_\bullet S_\bullet(\mathcal{C}^{\mathrm{small}})|$ is independent of this choice, up to homotopy equivalence.

In the case $\mathcal{C} = \mathrm{Proj}_{\mathbb{Z}}$ of finitely generated projective modules over $\mathbb{Z}$, we may make the following explicit choice of $(\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}}$. The set of objects of $(\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}}$ is $\mathbb{N} = \{0,1,2,\dots\}$. The set $\mathrm{Mor}_{(\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}}}(a,b)$ is the set of $b \times a$ integer matrices. Composition is given by matrix multiplication. This category may be further equipped with a symmetric monoidal structure corresponding to direct sum of $\mathbb{Z}$-modules: the product is given on objects by $m(a,b) = a+b$, and on morphisms by block sum of matrices. The associator

$$\begin{tikzcd} (\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}} \times (\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}} \times (\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & (\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}} \end{tikzcd}$$

is given on (a,b,c) by the identity matrix of size $a+b+c$. The symmetry natural isomorphism is given on $(a,b) \in (\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}} \times (\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}}$ by the $(a+b)\times(a+b)$ block matrix

$$\begin{pmatrix} 0 & \mathrm{Id}_a \\ \mathrm{Id}_b & 0 \end{pmatrix}.$$

In this case, sending $n \mapsto \mathbb{Z}^n$ extends to a symmetric monoidal equivalence from $(\mathrm{Proj}_{\mathbb{Z}})^{\mathrm{small}}$ to the category $\mathrm{Proj}_{\mathbb{Z}}$ of all finitely generated free $\mathbb{Z}$-modules (equipped with the cartesian symmetric monoidal structure, i.e., direct sum).

2.3. **Projective modules and K-theory.** Recall the category Proj_R of finitely generated projective left R-modules over a ring R, considered as a Waldhausen category as in Example 2.11. Then

$$BK(R) := |N_\bullet S_\bullet(\mathcal{C})|, \quad \text{and} \quad K_i(R) := \pi_{i+1} BK(R). \tag{14}$$

By (14) and its degree shift, $BK(R)$ is a de-looping of the K-theory space of R.

We now take $R = \mathbb{Z}$. Consider the following filtrations on the objects of $\mathrm{Proj}_{\mathbb{Z}}$ and $S_\bullet(\mathrm{Proj}_{\mathbb{Z}})$. First, an object P is in F_n if $\mathrm{rank}(P) \leq n$. This filtration induces a filtration on the objects of $S_p(\mathrm{Proj}_{\mathbb{Z}})$, in which an object of $S_p(\mathrm{Proj}_{\mathbb{Z}})$, which is a triangular diagram (13), is in F_n if the top right projective module is in F_n, i.e., has rank at most n. Since that rank is non-increasing under face and degeneracy maps, and is preserved under weak equivalences, it follows that the filtration on objects of $S_p(\mathrm{Proj}_{\mathbb{Z}})$, for each $p \geq 0$ yields a filtered bisimplicial set $N_\bullet S_\bullet(\mathrm{Proj}_{\mathbb{Z}})$. Hence, we obtain an exhaustive filtration of based spaces

$$F_0 BK(\mathbb{Z}) \subset \cdots \subset F_n BK(\mathbb{Z}) \subset F_{n+1} BK(\mathbb{Z}) \subset \cdots \subset BK(\mathbb{Z}), \tag{15}$$

where $F_0 BK(\mathbb{Z})$ is contractible.

Definition 2.14. *Let ${}^Q E^*_{*,*}$ denote the homology spectral sequence associated to the rank filtration on $BK(\mathbb{Z})$, called the* Quillen spectral sequence*.*

A filtration of $BK(\mathbb{Z})$ equivalent to the one above, and more generally a filtration of $BK(\mathcal{O}_F)$ for a number field F, was used by Quillen in [Qui73, pp. 179–199] for his proof that $K_i(\mathcal{O}_F)$ is a finitely generated abelian group for all $i \in \mathbb{N}$. Quillen's construction of the filtration used the "Q-construction" model for $BK(\mathcal{O}_F)$, and he also identified the associated graded with homology of the Steinberg module.

Proposition 2.15. *There is a canonical isomorphism*

$${}^Q E^1_{s,t} \cong H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s \otimes \mathbb{Q}) \Rightarrow H_*(BK(\mathbb{Z})). \tag{16}$$

With respect to this isomorphism, the algebra structure is induced by the block sum homomorphism $\mathrm{GL}_s(\mathbb{Z}) \times \mathrm{GL}_{s'}(\mathbb{Z}) \to \mathrm{GL}_{s+s'}(\mathbb{Z})$ and the $\mathrm{GL}_s(\mathbb{Z}) \times \mathrm{GL}_{s'}(\mathbb{Z})$-equivariant map $\mathrm{St}_s \otimes \mathrm{St}_{s'} \to \mathrm{St}_{s+s'}$ induced by the "block sum" map of spaces $T_s(\mathbb{Q}) * T_{s'}(\mathbb{Q}) \to T_{s+s'}(\mathbb{Q})$.

The Tits building $T_{s+s'}(\mathbb{Q})$ is the (nerve of the) poset of non-zero proper linear subspaces of $\mathbb{Q}^{s+s'} = \mathbb{Q}^s \oplus \mathbb{Q}^{s'}$. Identifying $\mathbb{Q}^s$ with $\mathbb{Q}^s \oplus \{0\} \subset \mathbb{Q}^{s+s'}$ and $\mathbb{Q}^{s'}$ with $\{0\} \oplus \mathbb{Q}^{s'} \subset \mathbb{Q}^{s+s'}$ leads to a canonical embedding of simplicial complexes

$$\begin{aligned} T_s(\mathbb{Q}) * T_{s'}(\mathbb{Q}) &\hookrightarrow T_{s+s'}(\mathbb{Q}) \\ (V \in T_s(\mathbb{Q})) &\mapsto V \oplus \{0\} \\ (V' \in T_{s'}(\mathbb{Q})) &\mapsto \{0\} \oplus V', \end{aligned}$$

and this is the "block sum" map referred to in the proposition. It is evidently equivariant with respect to the usual block sum operation of matrices

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let us also remark that for $s = 0$, one should take $\mathrm{St}_0 = \mathbb{Q}$ in the above formula for the E^1 page.

Proof. As mentioned above this is essentially due to Quillen, although his construction of the spectral sequence and identification of the E^1 page with the homology of the Steinberg module used a different model of $BK(\mathbb{Z})$. His model, the Q-construction, compares to the Waldhausen construction by an explicit homotopy equivalence explained in [Wal85, Section 1.9]. One may verify that this homotopy equivalence induces a homotopy equivalence in each filtration degree.

We will later, in Section 2.5, need to use what the isomorphism ${}^{Q}E^1_{s,t} \cong H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s \otimes \mathbb{Q})$ is, so we sketch a direct construction which makes no mention of the Q-construction. Let us write

$$S^n_p(\mathrm{Proj}_{\mathbb{Z}}) = F_n S_p(\mathrm{Proj}_{\mathbb{Z}}) \setminus F_{n-1} S_p(\mathrm{Proj}_{\mathbb{Z}})$$

for the full subgroupoid on those triangular diagrams (13) in which $P_{0,p} \cong \mathbb{Z}^n$. Then the filtration quotients are described, in bidegree (p, q), by the bijection of pointed sets

$$\frac{F_n N_q S_p(\mathrm{Proj}_{\mathbb{Z}})}{F_{n-1} N_q S_p(\mathrm{Proj}_{\mathbb{Z}})} \cong N_q S^n_p(\mathrm{Proj}_{\mathbb{Z}}) \cup \{\infty\}. \tag{17}$$

Here, the quotient S/T of a set S by a subset T, or more generally along a morphism $T \to S$, is understood as the pushout of the diagram $\{*\} \leftarrow T \to S$. This pushout S/T is naturally regarded as a pointed set with distinguished element corresponding to the image of $\{*\} \to S/T$. On the right hand side, ∞ denotes a disjoint basepoint corresponding to the collapsed subset. There are well defined functors $d_i \colon S^n_p(\mathrm{Proj}_{\mathbb{Z}}) \to S^n_{p-1}(\mathrm{Proj}_{\mathbb{Z}})$ for $0 < i < p$, defined by deleting the ith row and column of the triangular diagram, but the outer faces d_0 and d_p will not always land in the subgroupoid $S^n_{p-1}(\mathrm{Proj}_{\mathbb{Z}}) \subset F_n S_p(\mathrm{Proj}_{\mathbb{Z}})$ because the rank may drop (which happens unless $P_{0,1} \cong 0$ or $P_{p-1,p} \cong 0$, respectively). In the bisimplicial set (17), the face operators in the p-direction will send a simplex to the basepoint ∞ when this happens. We deduce that $E^1_{n,*}$ is calculated as the homology of the total complex associated to the double complex with

$$C_{p,q} = \mathbb{Q}\langle N_q S^n_p(\mathrm{Proj}_{\mathbb{Z}})\rangle = \frac{\mathbb{Q}\langle N_q S^n_p(\mathrm{Proj}_{\mathbb{Z}}) \cup \{\infty\}\rangle}{\mathbb{Q}\langle\{\infty\}\rangle},$$

where $\mathbb{Q}\langle X\rangle$ denotes the free $\mathbb{Q}$-module on a set X, and boundary maps defined as alternating sum of face maps as usual.

The groupoid $S_p(\mathrm{Proj}_{\mathbb{Z}})$ comes with a functor F to the groupoid of finitely generated projective $\mathbb{Z}$-modules and their isomorphisms, defined by sending a triangular diagram (13) to $P_{0,p}$. The categorical fiber $(F \downarrow \mathbb{Z}^n)$ over the object $\mathbb{Z}^n$ is the groupoid whose objects are triangular diagrams equipped with a specified isomorphism $P_{0,p} \to \mathbb{Z}^n$, and where morphisms from one triangular diagram to another is an isomorphism of diagrams compatible with the two specified isomorphisms from upper right entries to $\mathbb{Z}^n$. If we denote this groupoid $\widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}})$, then there is a forgetful map

$$N_q \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}}) \to N_q S^n_p(\mathrm{Proj}_{\mathbb{Z}}) \tag{18}$$

which identifies $N_q S^n_p(\mathrm{Proj}_{\mathbb{Z}})$ with the set of orbits for the free $\mathrm{GL}_n(\mathbb{Z})$-action on the set $N_q \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}})$, defined by acting by post-composition on the specified isomorphisms $P_{0,p} \cong \mathbb{Z}^n$. There is also a map of sets

$$N_q \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}}) \to \pi_0(N_\bullet \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}})) \tag{19}$$

sending a q-simplex to the path component containing it. Regarding p as fixed, the maps (19) assemble to a map of simplicial sets from $[q] \mapsto N_q \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}})$ to the constant simplicial set $[q] \mapsto \pi_0(N_\bullet \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}}))$. This map is a weak homotopy equivalence because automorphisms in the groupoid $\widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}})$ are uniquely determined by their action on the top-right module $P_{0,p} = \mathbb{Z}^n$ and hence all automorphism groups in this groupoid are trivial.

The set $\pi_0(N_\bullet \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}}))$ is in bijection with the set of flags $0 \subset P_{0,1} \subset \cdots \subset P_{0,p-1} \subset \mathbb{Z}^n$ of saturated submodules. Such a flag is non-degenerate as a p-simplex of $[p] \mapsto \pi_0(N_\bullet \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}}))$ if and only if none of the inclusions are equalities. For $p \geq 2$, this set of non-degenerate p-simplices is in bijection with the set of $(p-2)$-dimensional faces of the Tits building $T_n(\mathbb{Q})$. Writing

$$\widetilde{C}_{p,q} = \mathbb{Q}\langle N_q \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}})\rangle = \frac{\mathbb{Q}\langle N_q \widetilde{S}^n_p(\mathrm{Proj}_{\mathbb{Z}}) \cup \{\infty\}\rangle}{\mathbb{Q}\langle\{\infty\}\rangle},$$

we deduce that the maps

$$\mathrm{Tot}_p\, \widetilde{C}_{*,*} \to \widetilde{C}_{p-2}(T_n(\mathbb{Q}))$$

that are induced by (19) on $\widetilde{C}_{p,0}$ and zero on $\widetilde{C}_{p',p-p'}$ for $p' \neq p$ assemble to a quasi-isomorphism of chain complexes, and hence, for $n \geq 1$, an isomorphism

$$H_p(\mathrm{Tot}\, \widetilde{C}_{*,*}) \cong \begin{cases} \mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q} & \text{for } p = n \\ 0 & \text{otherwise.} \end{cases}$$

The maps (18) induce an isomorphism of double complexes

$$\mathbb{Q} \otimes_{\mathbb{Q}[\mathrm{GL}_n(\mathbb{Z})]} \widetilde{C}_{*,*} \xrightarrow{\cong} C_{*,*}$$

and hence an isomorphism of associated total complexes. Since $\mathrm{Tot}(\widetilde{C}_{*,*})$ is a complex of free $\mathbb{Q}[\mathrm{GL}_n(\mathbb{Z})]$-modules, we can use it to compute group homology as

$$\begin{aligned} H_p(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) &= \mathrm{Tor}_p^{\mathbb{Q}[\mathrm{GL}_n(\mathbb{Z})]}(\mathbb{Q}, \mathrm{St}_n(\mathbb{Q})) = H_{p-n}(\mathbb{Q} \otimes_{\mathbb{Q}[\mathrm{GL}_n(\mathbb{Z})]} \mathrm{Tot}\, \widetilde{C}_{*,*}) \\ &= H_{p-n}(\mathrm{Tot}\, C_{*,*}) = H_{p-n}(F_n(BK(\mathbb{Z})), F_{n-1}(BK(\mathbb{Z})); \mathbb{Q}), \end{aligned}$$

as required.

The assertion about the algebra structure follows by tracing isomorphisms, since direct sum of $\mathbb{Z}$-modules corresponds to block sum of matrices upon choosing an isomorphism to $\mathbb{Z}^n$ for some n. □

Remark 2.16. We have stated the theorem and proof above for the ring $R = \mathbb{Z}$, but the rank filtration evidently makes sense for any ring R, by considering the full subcategories $F_g(\mathrm{Proj}_R)$ on those projective modules which arise as summands of $R^{\oplus g}$.

The formula for the E^1 page generalizes to any Dedekind domain R, for instance $R = \mathcal{O}_E$ for a number field E, or if R is any field. If K is the fraction field of a Dedekind domain R, then saturated subspaces of a finitely generated projective R-module P of rank s are in bijection with linear subspaces of $P \otimes_R K \cong K^{\oplus s}$, and the Tits building $T(P \otimes_R K) \approx T_s(K)$ is defined as the nerve of the partially ordered set of non-zero proper linear subspaces of $P \otimes_R K$. The geometric realization of this partially ordered set has the homotopy type of a wedge of $(s-2)$-spheres by the Solomon–Tits theorem, and the Steinberg module can be defined as $\mathrm{St}(P \otimes_R K) = \widetilde{H}_{s-2}(|T(P \otimes_R K)|) \cong \mathrm{St}_s(K)$. The proof above goes through in this generality with only minor modifications, giving a Quillen spectral sequence of the form

$${}^Q E^1_{s,t} = \bigoplus_P H_t(\mathrm{GL}(P); \mathrm{St}(P \otimes_R K)) \Rightarrow H_{s+t}(BK(R))$$

where P ranges over a (finite) set of representatives for the isomorphism classes of projective modules P over R of rank s, for any Dedekind ring R. For a completely general ring we still have a spectral sequence associated to the rank filtration, but not a useful formula for its E^1 page.

In Section 5.1, we explain a construction of K-theory for graphs that is closely related to the $S_\bullet$-construction for Waldhausen categories. Several variants are possible. The variant we use, which seems most closely related to the Quillen spectral sequence, does not fit the definitions of a Waldhausen category, but is similar in spirit.

2.4. **Canonical forms for** GL_n**.** The space P_g of positive-definite real symmetric matrices of rank g is equipped with a left action of $\mathrm{GL}_g(\mathbb{R})$ given by $X \mapsto gXg^T$. Classical invariant theory provides a differential n-form for every $n \geq 1$

$$\omega^n_X = \mathrm{tr}((X^{-1}dX)^n)$$

which is invariant under the action of $\mathrm{GL}_g(\mathbb{R})$. One shows that ω^n vanishes unless $n \equiv 1 \pmod 4$; the case $n = 1$ plays very little role in what follows. The ω^n have a number of useful properties, including compatibility with block direct sums of matrices:

$$\omega^n_{X_1 \oplus X_2} = \omega^n_{X_1} + \omega^n_{X_2} \tag{20}$$

and, in the case when $g > 1$ is odd, the form ω^{2g-1} is singled out from its vanishing property

$$\omega^{2g-1}_X = 0 \qquad \text{if } X \text{ has rank } < g\ . \tag{21}$$

By invariance, the forms ω^{4k+1}_X for $k > 1$ define differential forms on the locally symmetric space $P_g/\mathrm{GL}_g(\mathbb{Z})$. Borel showed that the graded exterior algebra they generate is isomorphic to its stable cohomology:

$$\bigwedge \bigoplus_{k \geq 1} \omega^{4k+1}\mathbb{R} \ \xrightarrow{\ \sim\ }\ \varprojlim_g H^*(P_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R})\ .$$

Both sides of this equation have a Hopf algebra structure such that the (indecomposable) generators ω^{4k+1} are primitive. This follows from (20). By taking primitives, Borel deduced that $K_n(\mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q}$ for all $n = 4k+1$ with $k > 1$, and vanishes for all other $n > 0$.

We will need much more precise results about the cohomology of $\mathrm{GL}_n(\mathbb{Z})$ in the unstable range. For this, let $g > 1$ be odd and let $\Omega^*(g)$ denote the graded exterior algebra generated by $\omega^5, \ldots, \omega^{2g-1}$, which are the non-vanishing forms on $P_g/\mathrm{GL}_g(\mathbb{Z})$. It is a direct sum

$$\Omega^*(g) = \Omega^*_c(g) \oplus \Omega^*_{nc}(g)$$

where $\Omega^*_c(g)$ is the graded vector space of forms 'of compact type' given by the ideal generated by ω^{2g-1}, and $\Omega^*_{nc}(g)$ is the graded algebra generated by $\omega^5, \ldots, \omega^{2g-5}$. The Hodge star operator interchanges these two spaces. In [Bro23] it was shown that every element in $\Omega_c(g)$ defines a unique compactly supported cohomology class, giving two injective maps

$$\Omega^*_{nc}(g) \longrightarrow H^*(P_h/\mathrm{GL}_h(\mathbb{Z});\mathbb{R}) \qquad \text{for all } h \geq g, \tag{22}$$

$$\Omega^*_c(g)[-1] \longrightarrow H^*_c(P_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R})\ , \tag{23}$$

the first of which is a map of graded algebras, the second only a map of graded vector spaces. By taking the limit of the first injection, one obtains a stronger version of Borel's theorem (which states that (22) is injective in degrees $\leq g/4$ when $h = g$.) However, it is the second injective map (23) which is involved in a key definition.

Definition 2.17. *Set $\Omega_c(g) = 0$ for g even. Denote by*

$$\Omega_c^* = \bigoplus_{g>1} \Omega_c^*(g)$$

the $\mathbb{Q}$ vector space spanned by the canonical forms of compact type, bigraded as follows. The degree *and* genus *are given on generators by*

$$\begin{aligned} \deg(\omega^{4i_1+1} \wedge \cdots \wedge \omega^{4i_k+1}) &= \Sigma_{j=1}^k (4i_j + 1) \\ \mathrm{g}(\omega^{4i_1+1} \wedge \cdots \wedge \omega^{4i_k+1}) &= 2i_k + 1 \end{aligned}$$

for integers $0 < i_1 < \cdots < i_k$. Then a generator ω is in bidegree $(\mathrm{g}(\omega), \mathrm{g}(\omega) - \deg(\omega))$.

Although the bigraded vector space Ω_c^* is formally isomorphic to the space of elements of positive degree in the graded exterior algebra generated by the $\{\omega^{4k+1} : k \geq 1\}$, it is advisable not to confuse the two. The former is a bigraded vector space, while the latter is the positive degree elements of a Hopf algebra. Indeed, we shall prove that the image of Ω_c^* is primitive with respect to the coproduct we shall define, which is *not* the case for non-trivial products of ω^{4k+1} in the stable cohomology of the general linear group. The precise relationship between these two coproducts is via the Quillen spectral sequence, explained below.

2.4.1. *Canonical tensor algebra.* Denote the tensor algebra on a bigraded vector space V by

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} .$$

It is a connected, bigraded Hopf algebra with non-commutative product given by the tensor product, and (graded) cocommutative coproduct $\Delta : T(V) \to T(V) \otimes T(V)$ dual to the shuffle product, with respect to which the elements of V are primitive. We shall denote elements $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n} \subset T(V)$ using the bar notation $[v_1 | \ldots | v_n]$.

Let $\Omega_c^*[-1]$ denote the bigraded $\mathbb{Q}$-vector space in which degree is shifted by 1. So ω^{4i+1} has genus $2i + 1$ and degree $(4i + 1) + 1$. The tensor algebra $T(\Omega_c^*[-1])$ is a non-commutative, (graded) cocommutative Hopf algebra. It is again bigraded by genus and degree minus genus, where $[v_1 | \ldots | v_n]$ has degree $\sum \deg(v_i)$ and genus $\sum \mathrm{g}(v_i)$. Note that it has an additional grading by length of tensors, but only the associated filtration will play a role.

2.5. **Locally symmetric spaces and the Quillen spectral sequence.** Recall that the tropical spectral sequence and the Quillen spectral sequence have isomorphic E^1 pages

$${}^T E^1_{s,t} \cong H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s \otimes \mathbb{Q}) \cong {}^Q E^1_{s,t}.$$

The goal of this subsection is to make the implied isomorphism of E^1 pages more explicit. Recall that Borel–Moore homology of a reasonable space agrees with the relative homology of its one-point compactification, for instance $H_s^{\mathrm{BM}}(A_g^{\mathrm{trop}}) = H_s(A_g^{\mathrm{trop}} \cup \{\infty\}, \infty)$. Therefore, the tropical spectral sequence can be viewed as the spectral sequence associated (in ordinary singular homology) to the filtered space

$$A_\infty^{\mathrm{trop}} \cup \{\infty\} = \operatorname*{colim}_{g \to \infty} A_g^{\mathrm{trop}} \cup \{\infty\},$$

the direct limit of the one-point compactifications of the locally compact space A_g^{trop}. The associated graded has

$$\mathrm{Gr}_g(A_\infty^{\mathrm{trop}} \cup \{\infty\}) \cong (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\},$$

the one-point compactification of $P_g/\mathrm{GL}_g(\mathbb{Z})$.

In this subsection we will construct an explicit zig-zag of (rational) equivalences

$$\mathrm{Gr}_g(BK(\mathbb{Z})) \xleftarrow{\simeq} |N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \xrightarrow{\simeq_{\mathbb{Q}}} \mathrm{Gr}_g(A_\infty^{\mathrm{trop}} \cup \{\infty\})) \tag{24}$$

in which the middle term $|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}|$ will be defined in this section. The reduced homology of the rightmost space agrees with the Borel–Moore homology of $P_g/\mathrm{GL}_g(\mathbb{Z})$ and hence is the E^1 page of the tropical spectral sequence. The reduced homology of the leftmost space in the zig-zag is manifestly the E^1 page of the Quillen spectral sequence, and for both spectral sequences we have identified the E^1 page as $E^1_{s,t} = H_t(\mathrm{GL}_s(\mathbb{Z}), \mathrm{St}_s \otimes \mathbb{Q})$. The purpose of discussing this zig-zag is to evaluate certain pairings: given classes

$$w \in {}^Q E^1_{s,t} \cong H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s \otimes \mathbb{Q})$$
$$\omega \in {}^T E_1^{s,t} \otimes \mathbb{R} \cong \mathrm{Hom}(H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s), \mathbb{R})$$

we wish to make sense of the pairing $\langle w, \omega \rangle \in \mathbb{R}$. To do this we need to pin down the isomorphism ${}^Q E^1_{s,t} \to {}^T E^1_{s,t}$, which is what the zig-zag (24) is useful for. In practice, the homology class w will be represented by a somewhat explicit cycle in $\mathrm{Gr}_g(BK(\mathbb{Z}))$ arising from graphs, and the cohomology class ω will be given by an explicit $\mathrm{GL}_g(\mathbb{Z})$-equivariant differential form on P_g. In this situation, the strategy will be to transfer w along the zig-zag to get an interpretation as cycle in $(A_g^{\mathrm{trop}} \cup \{\infty\})/(A_{g-1}^{\mathrm{trop}} \cup \{\infty\})$ and evaluate the pairing by performing an integral.

Before proceeding with the construction of the zig-zag (24), let us point out that this must happen at the level of associated graded and cannot be lifted to the level of filtered spaces, since the tropical spectral sequence collapses to $E^2 = 0$ while the Quillen spectral sequence does not.

We first define the space in the middle.

Definition 2.18. *For a finite-dimensional $\mathbb{Q}$-vector space V, let $N_0 T_p(V)$ be the set of pairs $(A_p \subset \cdots \subset A_0, <)$ where*

$$A_p \subset \cdots \subset A_0 \subset V^\vee \setminus \{0\}$$

is a flag of non-zero vectors in the dual vector space, $<$ is a total order on A_0, satisfying the conditions

(1) A_i is finite for all i,
(2) $A_p = \emptyset$,
(3) $\mathrm{span}(A_0) = V^\vee$.

Let $N_0 T_p(V) \cup \{\infty\}$ denote the pointed set obtained by adding a disjoint base point, which we denote ∞, and define maps of pointed sets

$$d_i : N_0 T_p(V) \cup \{\infty\} \to N_0 T_{p-1}(V) \cup \{\infty\}$$

for $0 \leq i \leq p$ by the formula

$$d_i(A_p \subset \cdots \subset A_0, <) = \begin{cases} \infty & \textit{if } i = 0 < p \textit{ and } \mathrm{span}(A_1) \neq V^\vee \\ \infty & \textit{if } i = p > 0 \textit{ and } A_{p-1} \neq \emptyset \\ (A_p \subset \cdots \subset \widehat{A_i} \subset \cdots \subset A_0, <) & \textit{otherwise.} \end{cases}$$

There are also degeneracy maps $s_i : N_0T_{p-1}(V) \cup \{\infty\} \to N_0T_p(V) \cup \{\infty\}$ defined by duplicating A_i, making $[p] \mapsto N_0T_p(V) \cup \{\infty\}$ into a simplicial object in the category of pointed sets.

There is an evident action of $\mathrm{GL}_g(\mathbb{Z})$ on $N_0T_p(\mathbb{Q}^g)$, which we can use to turn $N_0T_p(\mathbb{Q}^g)$ into the objects of a groupoid $T_p(\mathbb{Q}^g)$: the set of morphisms from $(A_p \subset \cdots \subset A_0)$ to $(A'_p \subset \cdots \subset A'_0)$ is the set of matrices $X \in \mathrm{GL}_g(\mathbb{Z})$ with the property that $X.(A_p \subset \cdots \subset A_0) = (A'_p \subset \cdots \subset A'_0)$. (This action is of course the restriction of an action of $\mathrm{GL}_g(\mathbb{Q})$, but for our purposes we need the action by integer matrices only.) Explicitly,

$$N_qT_p(\mathbb{Q}^g) = (\mathrm{GL}_g(\mathbb{Z}))^q \times N_0T_p(\mathbb{Q}^g),$$

with face maps in the q-direction given by

$$d_i(X_1, \dots, X_q, (A_p \subset \cdots \subset A_0, <)) = \begin{cases} (X_2, \dots, X_q, (A_p \subset \cdots \subset A_0, <)) & \textit{for } i = 0 \\ (X_1, \dots, X_iX_{i+1}, \dots, X_q, (A_p \subset \cdots \subset A_0, <)) & \textit{for } 0 < i < p \\ (X_1, \dots, X_{q-1}, X_q.(A_p \subset \cdots \subset A_0, <)) & \end{cases}$$

The notation $N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}$ should be read as the bisimplicial pointed set

$$([p], [q]) \mapsto N_qT_p(\mathbb{Q}^g) \sqcup \{\infty\},$$

with face and degeneracy operators as defined above.

Remark 2.19. All automorphism groups in the groupoid $T_p(\mathbb{Q}^g)$ are trivial, so the canonical map

$$|N_\bullet T_p(\mathbb{Q}^g)| \to \pi_0(N_\bullet T_p(\mathbb{Q}^g)) \cong N_0(T_p(\mathbb{Q}^g))/\mathrm{GL}_g(\mathbb{Z}) \tag{25}$$

is a weak equivalence for all p. Therefore the middle space in the zig-zag (24) is also homotopy equivalent to the geometric realization of the simplicial set

$$[p] \mapsto N_0T_p(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}.$$

We need both simplicial directions for the map to $\mathrm{Gr}_g(BK(\mathbb{Z}))$ though.

We first explain how to construct the map $|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to \mathrm{Gr}_g(BK(\mathbb{Z}))$ in the zig-zag (24). Here it is natural to look for a map of bisimplicial sets, with (p, q)-simplices

$$N_qT_p(\mathbb{Q}^g) \cup \{\infty\} \to \mathrm{Gr}_g(N_qS_p(\mathrm{Proj}_\mathbb{Z})).$$

We explain how this works for $q = 0$, starting with $(\emptyset = A_p \subset \cdots \subset A_0, <) \in N_0T_p(V)$.

The subsets $A_i \subset V^\vee \setminus \{0\}$ give rise to canonical maps

$$\begin{aligned} V &\to \mathbb{Q}^{A_i} \\ v &\mapsto (\psi \mapsto \psi(v)), \end{aligned}$$

which by assumption is injective for $i = 0$. For $V = \mathbb{Q}^g$ we shall make use of the restrictions

$$\mathbb{Z}^g \hookrightarrow \mathbb{Q}^g \to \mathbb{Q}^{A_i}$$

in the following. For $0 \le i \le j \le p$, define a free $\mathbb{Z}$-module $P_{i,j}$ as the image of the composition

$$\mathrm{Ker}(\mathbb{Z}^g \to \mathbb{Q}^{A_j}) \hookrightarrow \mathbb{Z}^g \to \mathbb{Q}^{A_i}.$$

These modules fit into a triangular diagram of the form (13), forming an object of $S_p(\mathrm{Proj}_{\mathbb{Z}})$. Let us also remark that for $j = p$, the submodule $P_{i,p} \subset \mathbb{Q}^{A_i}$ agrees with the image of the second map $\mathbb{Z}^g \to \mathbb{Q}^{A_i}$, which for $i = 0$ is injective. Hence we obtain a canonical isomorphism $\mathbb{Z}^g \to P_{0,p}$. We have constructed a map of sets

$$N_0 T_p(\mathbb{Q}^g) \to F_g(N_0 S_p(\mathrm{Proj}_{\mathbb{Z}})) \setminus F_{g-1}(N_0 S_p(\mathrm{Proj}_{\mathbb{Z}})), \tag{26}$$

which is easily extended to a map of bisimplicial pointed sets

$$N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\} \to \mathrm{Gr}_g(N_\bullet S_\bullet(\mathrm{Proj}_{\mathbb{Z}})),$$

which in turn realizes to the desired map of spaces

$$|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to \mathrm{Gr}_g(BK(\mathbb{Z})).$$

Proposition 2.20. *This map $|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to \mathrm{Gr}_g(BK(\mathbb{Z}))$ induces an isomorphism in integral homology.*

Proof. The simplicial set $N_0 T_\bullet(V) \cup \{\infty\}$ has a cubical structure (similar in nature to a space $|\mathcal{F}_\bullet|$ which we will define later), as follows. For each pair $(A, <)$ consisting of a finite subset $A \subset V^\vee \setminus \{0\}$ and a total order $<$ on A, the set of elements $(A_p \subset \cdots \subset A_0, <') \in N_0 T_p(V) \cup \{\infty\}$ for which $A_0 \subset A$ and $<'$ is the restriction of $<$, assemble as p varies to a map of simplicial sets

$$(\Delta^1_\bullet)^A \xrightarrow{(A,<)} N_0 T_\bullet(V) \cup \{\infty\}.$$

This map lands in the basepoint ∞ unless A spans $V^\vee$, in which case the induced map of geometric realizations

$$|\Delta^1_\bullet|^A \xrightarrow{|(A,<)|} |N_0 T_\bullet(V)| \cup \{\infty\} \tag{27}$$

is injective when restricted to $(\Delta^1 \setminus \partial\Delta^1)^A$. As $(A, <)$ varies over all possible (totally ordered) finite spanning sets of $V^\vee$, the maps (27) together with the basepoint ∞ form a CW structure on the space $|N_0 T_\bullet(V) \cup \{\infty\}|$. It follows that the reduced homology of $|N_0 T_\bullet(V) \cup \{\infty\}|$ may be computed by a chain complex defined combinatorially by generators $[\phi_0, \ldots, \phi_p] \in (V^\vee \setminus \{0\})^{p+1}$ for varying p, subject to the relation that $[\phi_0, \ldots, \phi_p] = 0$ unless the ϕ_i span $V^\vee$. (Notice that we do not impose any relations between generators that differ by the action of the symmetric group S_{p+1} on the set of ordered $(p + 1)$-tuples.) The boundary map is given by

$$\partial[\phi_0, \ldots, \phi_p] = \sum_{i=0}^{p} (-1)^i [\phi_0, \ldots, \widehat{\phi_i}, \ldots, \phi_p].$$

For $V = \mathbb{Q}^g$, this chain complex is in fact a well known free resolution of $\mathrm{St}_g(\mathbb{Q})$ as a module over $\mathrm{GL}_g(\mathbb{Q})$ due to Lee and Szczarba [LS76, Section 3], and we deduce

$$\widetilde{H}_s(|N_0 T_\bullet(\mathbb{Q}^g) \cup \{\infty\}|) \cong \begin{cases} \mathrm{St}_g(\mathbb{Q}) & \text{for } p = g \\ 0 & \text{otherwise.} \end{cases}$$

The Steinberg module $\mathrm{St}_g(\mathbb{Q})$ is also the (reduced) homology of $N_\bullet \widetilde{S}^g_\bullet(\mathrm{Proj}_{\mathbb{Z}}) \cup \{\infty\}$, the bisimplicial set from the proof of Proposition 2.15 above. It is not hard to compare these two instances of the Steinberg module. Indeed, the map (26) was constructed by a recipe in which the "upper right" module $P_{0,p}$ in the triangular diagram came with a preferred isomorphism $\mathbb{Z}^g \to P_{0,p}$, and this provides a lift to

$$N_0 T_\bullet(\mathbb{Q}^g) \cup \{\infty\} \to N_0 \widetilde{S}^g_\bullet(\mathrm{Proj}_\mathbb{Z}) \cup \{\infty\}.$$

On reduced homology, the composition

$$|N_0 T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to |N_0 \widetilde{S}^g_\bullet(\mathrm{Proj}_\mathbb{Z}) \cup \{\infty\}| \to |N_\bullet \widetilde{S}^g_\bullet(\mathrm{Proj}_\mathbb{Z}) \cup \{\infty\}|$$

therefore induces a map between two modules isomorphic to $\mathrm{St}_g(\mathbb{Q})$. It is the same map as considered by Lee and Szczarba and hence an isomorphism.

By a spectral sequence argument, it follows that the map

$$|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to |N_\bullet S^g_\bullet(\mathrm{Proj}_\mathbb{Z})| \cup \{\infty\}$$

also induces an isomorphism on homology. □

Next we define the map to $\mathrm{Gr}_g(A^{\mathrm{trop}}_\infty \cup \{\infty\}) = (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}$ in (24) as a composition

$$|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \xrightarrow{\simeq} |N_0 T_\bullet(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}| \to \mathrm{Gr}_g(A^{\mathrm{trop}}_\infty \cup \{\infty\}),$$

where the first map is induced by (25) above, and we must define the second map. To an object $(\emptyset = A_p \subset \cdots \subset A_0, <) \in N_0 T_p(\mathbb{Q}^g)$ we associate a map $\Delta^g \to (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}$, constructed in the following way. For $(\emptyset = A_p \subset \cdots \subset A_0, <) \in N_0 T_p(\mathbb{Q}^n)$ consider the map

$$\begin{aligned} (\Delta^p \setminus \partial\Delta^p) &\to P_g \\ (0 < s_1 < \cdots < s_p < 1) &\mapsto \sum_{i=1}^{p} (-\log(s_i)) \sum_{\psi \in A_{i-1} \setminus A_i} \psi^2, \end{aligned} \tag{28}$$

where $\psi^2 \in P^{\mathrm{rt}}_g$ denotes the rank-1 quadratic form $v \mapsto (\psi(v))^2$. In this formula, the function $s \mapsto -\log(s)$ may be replaced by any other homeomorphism $(0,1) \to (0,\infty)$. Letting $(A_p \subset \cdots \subset A_0, <) \in N_0 T_p(\mathbb{Q}^g)$ vary, these maps assemble to a $\mathrm{GL}_g(\mathbb{Q})$-equivariant map

$$\coprod_p (\Delta^p \setminus \partial\Delta^p) \times N_0 T_p(\mathbb{Q}^g) \to P_g.$$

Passing to orbit sets for the action of the subgroup $\mathrm{GL}_g(\mathbb{Z})$, we obtain a map which extends as in the following diagram

$$\begin{tikzcd} \coprod_p (\Delta^p \setminus \partial\Delta^p) \times N_0 T_p(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z}) \arrow[r] \arrow[d, hook] & P_g/\mathrm{GL}_g(\mathbb{Z}) \arrow[d] \\ |N_0 T_\bullet(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}| \arrow[r] & (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}. \end{tikzcd} \tag{29}$$

Proposition 2.21. *The bottom horizontal map in (29) induces an isomorphism in reduced rational homology.*

Proof. The homology of the domain is isomorphic to ${}^Q E^1_{g,*} = H_*(\mathrm{GL}_g(\mathbb{Z}); \mathrm{St}_g \otimes \mathbb{Q})$, as is the homology of the codomain by duality (see §2.1.1). Therefore the rational homology groups of domain and codomain are abstractly isomorphic. Since the homology groups are also finite-dimensional, it suffices to construct a one-sided inverse to the map.

To produce a one-sided inverse (on the level of rational homology), the strategy will be to choose a suitable "simplicial structure" on $P_g/\mathrm{GL}_g(\mathbb{Z})\cup\{\infty\}$, namely a simplicial pointed set $X_\bullet$ with finitely many non-degenerate simplices, and a homotopy equivalence $|X_\bullet| \to P_g/\mathrm{GL}_g(\mathbb{Z})\cup\{\infty\}$, and then defining the one-sided inverse on the simplicial chains of $X_\bullet$ by sending each non-degenerate non-basepoint simplex to some preferred lift in $N_0T_\bullet(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}$. We will not quite succeed in doing exactly this, but can achieve that simplices of $X_\bullet$ come with a finite set of specified lifts. We then produce the one-sided inverse by taking average of those lifts.

To construct the simplicial set $X_\bullet$, we first choose an admissible decomposition of P_g^{rt} whose rays are all rank 1, for instance the perfect cone decomposition. Each cone σ is then the convex hull of its extremal rays $\mathbb{R}_{\geq 0} \to P_g^{\mathrm{rt}}$ which are of the form $t \mapsto t\psi^2$ for some $\psi \in (\mathbb{Q}^g)^\vee \setminus \{0\}$. Up to scaling by a non-zero rational number we can arrange that ψ is *normalized* to satisfy $\psi(\mathbb{Z}^g) = \mathbb{Z}$, and for normalized ψ the form $\psi^2 \in P_g^{\mathrm{rt}}$ is uniquely determined by the projective class $[\psi] \in \mathbb{P}((\mathbb{Q}^g)^\vee)$, because ψ itself is determined up to a sign by the normalization condition.

Any cone in the cone decomposition is then the image of a map of the form

$$\begin{aligned} \mathbb{R}^p_{\geq 0} &\to P_g^{\mathrm{rt}} \\ (t_1, \dots, t_p) &\mapsto \sum_{i=1}^p t_i \psi_i^2 \end{aligned}$$

for some normalized $\psi_i \in (\mathbb{Q}^g)^\vee \setminus \{0\}$. The map need not be injective unless the cone is simplicial, but it will always be a proper homotopy equivalence onto the cone (the inverse image of any point in the cone is a compact and convex subset of the octant). If the ψ_i span $(\mathbb{Q}^g)^\vee$, then the open cone has image in P_g and is also the image of the map

$$\begin{aligned} (\Delta^1 \setminus \partial\Delta^1)^p &\to P_g \\ (s_1, \dots, s_p) &\mapsto \sum_{i=1}^p (-\log s_i)\psi_i^2 \end{aligned}$$

analogous to (28) but without inequalities among the s_i. The admissible decomposition gives finitely many $\mathrm{GL}_g(\mathbb{Z})$-orbits of such maps, whose image in $P_g/\mathrm{GL}_g(\mathbb{Z})$ forms a decomposition into open cones (which may or may not be simplicial). After 1-point compactifying, we obtain finitely many maps of the form

$$|\Delta^1_\bullet|^p \to (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}, \tag{30}$$

one for each $\mathrm{GL}_g(\mathbb{Z})$-orbit of cones spanned by p many rays $t \mapsto t\psi_i^2$, with the property that the ψ_i span $(\mathbb{Q}^g)^\vee$. The image of each such map is the closure of the corresponding cone, but the map need not be injective. However, the inverse image of a point in the cone is contractible. If we pre-compose with the geometric realization of any of the top-dimensional simplices $\Delta^p_\bullet \to (\Delta^1_\bullet)^p$ then we obtain $p!$ many maps of the form (28) for each cone.

The maps $\Delta^p \to P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}$ arising this way, $p!$ many for each cone in the decomposition with p many extremal rays, can be regarded as elements of the pointed set $\mathrm{Sin}_p((P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\})$. Together with all iterated faces and degeneracies of these elements, and with the basepoint, they form a simplicial subset

$$X_\bullet \subset \mathrm{Sin}_\bullet((P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\})$$

with the property that the induced map

$$|X_\bullet| \to P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}$$

has contractible fibers and that X has only finitely many non-degenerate simplices. It follows from the Vietoris–Begle theorem that this map induces an isomorphism on homology. The simplicial set $X_\bullet$ has only finitely many non-degenerate simplices, on each of which the map is of the form (28) and therefore agrees with the map $|N_0T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}$ restricted to some p-simplex.

If we write $\widetilde{C}_*(X) = C_*(X_\bullet, \{\infty\}; \mathbb{Q})$ for the normalized rational chains relative to the base point, then we obtain a quasi-isomorphism

$$\widetilde{C}_*(X_\bullet) \xrightarrow{\simeq} \widetilde{C}_*(\mathrm{Sin}_\bullet(P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\})) \simeq \widetilde{C}^{\mathrm{sing}}_*(P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}; \mathbb{Q})$$

from a finite chain complex $\widetilde{C}_*(X_\bullet)$ similar in spirit to the "Voronoi complexes" of [EVGS13]. The main difference is that each cone in the decomposition, of geometric dimension d and spanned by p many extremal rays, gives rise to one generator of homological degree d in the Voronoi complex while in our complex it gives rise to $p!$ many generators of homological degree p and also some lower-dimensional generators.

Unfortunately, this does not quite identify $X_\bullet$ with a simplicial subset of $N_0T_\bullet(\mathbb{Q}^g) \cup \{\infty\}$, for two reasons. Firstly, the extremal rays of the cones in a cone decomposition are of the form $t \mapsto t\psi^2$ for some covector $\psi \in (\mathbb{Q}^g)^\vee$ but ψ is not quite canonically determined by the ray, only up to a sign. Secondly, elements of $N_0T_p(\mathbb{Q}^g)$ involve a total order $<$ which we cannot canonically produce from a p-simplex of $X_\bullet$. Altogether, we have explained a recipe by which a non-degenerate non-basepoint $\sigma \in X_p$ lifts in $2^p p!$ many ways to an element $\widetilde{\sigma} \in N_0T_p(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z})$. This is sufficient for defining a one-sided inverse on the level of rational homology though: let the diagonal map in the diagram

$$\begin{tikzcd}
 & \widetilde{C}^{\mathrm{sing}}_*(|N_0T_\bullet(\mathbb{Q}^g)/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}|; \mathbb{Q}) \arrow[d] \\
\widetilde{C}_*(X_\bullet) \arrow[ur] \arrow[r, "\simeq"] & \widetilde{C}^{\mathrm{sing}}_*(P_g/\mathrm{GL}_g(\mathbb{Z}) \cup \{\infty\}; \mathbb{Q})
\end{tikzcd}$$

be defined by sending a non-degenerate non-basepoint simplex of $X_\bullet$ to the average of the $2^p p!$ many lifts to $N_0T_p(\mathbb{Q}^g)$ that we have explained. ☐

Example 2.22. For a directed graph G with $b_1(G) = g$ the homology group $H_1(G)$ is a lattice in the vector space $V = H_1(G; \mathbb{Q})$. There is a canonical surjection

$$\mathbb{Q}^{E(G)} = C^1(G; \mathbb{Q}) \twoheadrightarrow H^1(G; \mathbb{Q}) = V^\vee$$

which sends the basis vector corresponding to $e \in E(G)$ to a non-zero vector in $V^\vee$ provided e is not a bridge in G. Let us assume G has no bridges.

Hence any flag of subsets $(\emptyset = A_p \subset \cdots \subset A_0 = E(G))$, together with a total order on $E(G)$, gives rise to an element of $N_0T_p(V)$ in the notation above. If we weaken the requirement $A_0 = E(G)$ to $A_0 \subset E(G)$ then we get an element of $N_0T_p(V) \cup \{\infty\}$. Letting p vary, these assemble to a map of simplicial sets

$$(\Delta^1_\bullet)^{E(G)} \to N_0T_\bullet(V) \cup \{\infty\}.$$

The total order on $E(G)$ also gives a fundamental chain $\iota \in C_{|E(G)|}((\Delta^1)^{E(G)};\mathbb{Z})$ which can be pushed forward to a chain $[G,\omega] \in C_{|E(G)|}(F_g BK(\mathbb{Z}), F_{g-1}BK(\mathbb{Z}))$. This association will later be upgraded to a map of spectral sequences of Hopf algebras, from a "graph spectral sequence" to the Quillen spectral sequence.

The fundamental cycle $\iota \in C_{|E(G)|}((\Delta^1)^n;\mathbb{Z})$ may also be pushed forward along the composition

$$(\Delta^1)^{E(G)} \to |N_0 T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}$$

representing a fundamental cycle of the image of this map. For the purposes of integrating differential forms on $P_g/\mathrm{GL}_g(\mathbb{Z})$ along this map, we may reparametrize using the diffeomorphism $s \mapsto -\log(s)$ to get a map

$$(0,\infty)^{E(G)} \to P_g/\mathrm{GL}_g(\mathbb{Z})$$

which can be identified with the *graph Laplacian* from [BS24].

Remark 2.23. Orthogonal direct sum of symmetric forms induces a map $P_g \times P_h \to P_{g+h}$. Passing to orbits leads to a map

$$(P_g/\mathrm{GL}_g(\mathbb{Z})) \times (P_h/\mathrm{GL}_h(\mathbb{Z})) \to (P_{g+h}/\mathrm{GL}_{g+h}(\mathbb{Z}))$$

which is proper. Therefore it induces a map of one-point compactifications, giving a product on ${}^T E^1_{*,*}$, the E^1 page of the tropical spectral sequence. The isomorphism

$${}^T E^1 \xrightarrow{\cong} {}^Q E^1$$

constructed above is an isomorphism of bigraded algebras with respect to this product on ${}^T E^1_{*,*}$ and the product on ${}^Q E^1_{*,*}$ constructed in the next section. (In the next section we also produce a coproduct, of which it seems more difficult to give a simple interpretation in ${}^T E$.)

3. Coproducts and filtrations in the Waldhausen construction

Recall that the rational homology of an infinite loop space such as $BK(\mathbb{Z})$ is naturally a graded commutative and cocommutative Hopf algebra, with coproduct given by push-forward under the diagonal. However, this coproduct does not respect the filtration on $BK(\mathbb{Z})$ and hence does not give rise to a coproduct on the pages of the Quillen spectral sequence.

Our goal in this section is to construct a *filtered coproduct* on $BK(\mathbb{Z})$. We do so via the simplicial operation of edgewise subdivision from [BHM93, Lemma 1.1]. The resulting coproduct induces a bigraded associative (but not cocommutative) coproduct on each page of the Quillen spectral sequence. It is homotopic to the diagonal and hence induces the usual commutative coproduct on the abutment.

3.1. **A filtered coproduct from edgewise subdivision.**

The goal now is to define a map

$$BK(\mathbb{Z}) \to BK(\mathbb{Z}) \times BK(\mathbb{Z})$$

that respects the filtration (15), i.e., with the property that the restriction to the nth filtration $F_n = F_n BK(\mathbb{Z})$ factors as

$$F_n \to \bigcup_{a+b=n} F_a \times F_b$$

so that it induces a coproduct on the associated relative homology spectral sequence. We will use the *edgewise subdivision* of [BHM93, Section 1], in turn inspired by [Seg73]. We recall the definition.

Definition 3.1. *For any category $\mathcal{S}$, given a simplicial object $X\colon \Delta^{\mathrm{op}} \to \mathcal{S}$, the edgewise subdivision* $\mathrm{es}(X)$ *of X is the simplicial object in $\mathcal{S}$*

$$\Delta \to \Delta \xrightarrow{X} \mathcal{S}$$

where the functor $\Delta \to \Delta$ is

$$[p] \mapsto [p] \sqcup [p].$$

Here, the set $[p] \sqcup [p]$ is linearly ordered by concatenation.

In other words, we identify $[p] \sqcup [p] \cong [1] \times [p]$ with the lexicographic ordering.

Remark 3.2. Note, in particular, that $\mathrm{es}_p(X) = X_{2p+1}$ for each $p \geq 0$.

For a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathsf{Set}$, there is a natural homeomorphism

$$|\mathrm{es}(X)| \xrightarrow{\phi_X} |X|.$$

It is the unique natural transformation that is affine on simplices and sends each vertex $x \in \mathrm{es}_0(X) = X_1$ to the mid-point of the corresponding edge or vertex (degenerate 1-simplex) in $|X|$. To check that this is a homeomorphism it suffices to consider the case where $X = \Delta(-,[n])$ is a simplex, since both sides preserve colimits. See [BHM93, Lemma 1.1].

We now consider how the homeomorphism ϕ_X and its inverse interact with filtrations.

Lemma 3.3. *Let $X_\bullet$ be a filtered simplicial set, i.e., a simplicial set X and a sequence of subsets $F_0X_p \subset F_1X_p \subset \dots$ for each p such that each $F_nX_\bullet$ forms a simplicial subset, i.e. $d_iF_nX_{p+1} \subset F_nX_p$ and $s_iF_nX_p \subset F_nX_{p+1}$ for all p and all i. For any filtered simplicial space, give the geometric realization $|X|$ the induced filtration: $F_n|X|$ is the image of the realization of the inclusion $|F_nX| \to |X|$. Finally, give the simplicial set $\mathrm{es}_\bullet(X)$ the filtration with $F_n\mathrm{es}_p(X) = \mathrm{es}_p(F_nX_\bullet) = F_nX_{2p+1}$.*

Then the inverse homeomorphism

$$|X| \xrightarrow{\phi_X^{-1}} |\mathrm{es}(X)|$$

is a filtered map: it satisfies $\phi_X^{-1}(F_n|X|) \subset F_n|\mathrm{es}(X)|$.

Proof. Let $f : X_p \to \mathbb{N}$ be the function such that $f(\sigma) = n$ if $\sigma \in F_nX_p \setminus F_{n-1}X_p$. Then each $F_nX_\bullet$ being a simplicial subset implies that $f(\theta^*\sigma) \leq f(\sigma)$ for any morphism $\theta : [p] \to [q]$ in Δ and any $\sigma \in X_q$, but then the simplicial identity $d_i \circ s_i(\sigma) = \sigma$ implies $f(s_i\sigma) = f(\sigma)$. In other words, the function f is determined by its values on non-degenerate simplices.

Now a point $x \in |X|$ will be in the image of the canonical injection

$$\{\sigma_x\} \times (\Delta^p \setminus \partial\Delta^p) \hookrightarrow |X|$$

for a unique $p \in \mathbb{N}$ and a unique non-degenerate $\sigma_x \in X_p$. Then $\phi_X^{-1}\colon |X| \xrightarrow{\sim} |\mathrm{es}(X)|$ sends x to a point in the image of the canonical injection

$$\{\tau_x\} \times (\Delta^q \setminus \partial\Delta^q) \hookrightarrow |\mathrm{es}(X)|$$

for a unique $q \in \{0,\dots,p\}$ and a unique $\tau_x \in \mathrm{es}_q(X) = X_{2q+1}$.

By naturality of the homeomorphism, there must exist a morphism $\theta_x : [p] \to [2q+1]$ in Δ such that $\theta_x^*(\sigma_x) = \tau_x$. This implies $f(\tau_x) \leq f(\sigma_x)$, and hence

$$f(\phi_X^{-1}(x)) \leq f(x)$$

as desired. □

Let us briefly discuss edgewise subdivision of bisimplicial sets $X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$, for which we write $X_{p,q} = X([p],[q])$. The notion itself is symmetric in p and q, but for the applications we have in mind the two simplicial directions will play quite different roles. In particular, when $\mathcal{C}$ is a Waldhausen category we will consider the bisimplicial set with

$$X_{p,q} = N_q S_p(\mathcal{C})$$

and use the *first* simplicial direction p as the "$S_\bullet$-direction". In this and other examples we consider, the second simplicial direction will play a more auxiliary role, mainly as a means to encode the simplicial topological space $[p] \mapsto |X_{p,\bullet}|$, while in the p-direction we will make more use of the simplicial structure. In particular, we will write

$$\mathrm{es}(X) = \mathrm{es}([p] \mapsto X_{p,\bullet})$$

for edgewise subdivision in the p-direction. In other words, on objects $[p]$ and $[q]$ of Δ we have

$$\mathrm{es}(X)_{p,q} = X_{2p+1,q} = X([p] \sqcup [p], [q]).$$

Proposition 3.4. *Let $X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$ be a bisimplicial set. Then $|\mathrm{es}(X)|$ is naturally homeomorphic to $|X|$.*

Proof. We have

$$|X| \cong |[q] \mapsto ([p] \mapsto X_{p,q})| \cong |[q] \mapsto ([p] \mapsto X_{2p+1,q})| \cong |[p] \mapsto ([q] \mapsto X_{2p+1,q})| \cong |\mathrm{es}(X)|,$$

using the fact that edgewise subdivision is a natural homeomorphism for simplicial sets, and that the two different geometric realizations of bisimplicial sets explained above are naturally homeomorphic. All of the other intermediate homeomorphisms are natural as well. □

Here is the main definition.

Definition 3.5. *Let $\mathcal{S}$ be a category with finite products. Let*

$$X\colon \Delta^{\mathrm{op}} \to \mathcal{S}$$

be a simplicial object in $\mathcal{S}$, and let $\mathrm{es}(X)$ be its edgewise subdivision. Define a natural transformation

$$\mathrm{es}(X) \Rightarrow X \times X \tag{31}$$

whose component at $[p]$, for each $p \geq 0$,

$$\mathrm{es}(X)_p = X_{2p+1} \to X_p \times X_p,$$

is given by the two maps $X_{2p+1} \to X_p$ induced by the two order preserving inclusions

$$[p] \to [p] \sqcup [p] \cong [2p+1]$$

onto the first $p+1$, respectively the last $p+1$, elements.

Remark 3.6. When $\mathcal{S}$ is the category of simplicial sets, $X : \Delta^{\mathrm{op}} \to \mathcal{S}$ is a bisimplicial set. Using that geometric realization commutes with finite products, the natural transformation $\mathrm{es}(X) \Rightarrow X \times X$ defined above leads to a map

$$|X| \xrightarrow{\simeq} |\mathrm{es}(X)| \to |X| \times |X|. \tag{32}$$

As a map of spaces, this map is not especially interesting by itself—as we will see momentarily, it is naturally homotopic to the diagonal map $x \mapsto (x, x)$ of the space $|X|$. What we will see is that in interesting examples, the bisimplicial set X comes with a filtration making (32) into a map of filtered spaces and (32) will induce an interesting map of associated gradeds, while the actual diagonal map of $|X|$ will not be a filtered map.

Example 3.7. Our main example comes from the simplicial object $X = S_\bullet(\mathrm{Proj}_\mathbb{Z})\colon [p] \mapsto S_p(\mathrm{Proj}_\mathbb{Z})$ in small groupoids, for the Waldhausen category $\mathrm{Proj}_\mathbb{Z}$ of finitely generated projective $\mathbb{Z}$-modules (Example 2.11). In this case the functor

$$\mathrm{es}_p(S_\bullet(\mathrm{Proj}_\mathbb{Z})) \to S_p(\mathrm{Proj}_\mathbb{Z}) \times S_p(\mathrm{Proj}_\mathbb{Z})$$

is given on objects by

$$(0 \to V_1 \to \cdots \to V_{2p+1}) \mapsto ((0 \to V_1 \to \cdots \to V_p), (0 \to V_{p+2}/V_{p+1} \to \cdots \to V_{2p+1}/V_{p+1})),$$

omitting the chosen quotients V_i/V_j from the notation.

We recall the filtration on $X = S_\bullet(\mathrm{Proj}_\mathbb{Z})$ described previously in (15). For integers p, q, an element of $X_{p,q}$ is a chain of q morphisms of triangular diagrams of projective modules of the form (13). The morphisms of triangular diagrams are required to be isomorphisms in each component. Define the rank of such an element to be the rank of the top-right projective $\mathbb{Z}$-module in any of the $q+1$ triangular diagrams; this rank is well-defined since the $q+1$ modules involved are related by isomorphisms. For any n, the property of having rank at most n is preserved by all face and degeneracy maps of X, and thus rank induces a filtered bisimplicial set in which

$$(F_n X)_{p,q} = X_{p,q} \cap \mathrm{rank}^{-1}((-\infty, n]).$$

The rank function on simplices of X induces a rank function on the simplices of $\mathrm{es}(X)$, as well as a rank function on the simplices of $X \times X$ by additivity: the rank of $(x, y) \in (X \times X)_{p,q} = X_{p,q} \times X_{p,q}$ is $\mathrm{rank}(x) + \mathrm{rank}(y)$. These rank functions are compatible with face and degeneracy maps, inducing filtrations on both $\mathrm{es}(X)$ and $X \times X$. We note that (31) respects the filtration, since

$$\mathrm{rank}(V_{2p+1}) \geq \mathrm{rank}(V_p) + \mathrm{rank}(V_{2p+1}/V_{p+1}). \tag{33}$$

Therefore, the natural transformation $\mathrm{es}(X) \Rightarrow X \times X$ is a map of filtered bisimplicial sets, and geometric realization yields a map

$$|X| \xrightarrow{\cong} |\mathrm{es}(X)| \longrightarrow |X \times X| \cong |X| \times |X|$$

of filtered spaces.

In Proposition 3.13, we shall study a product map $m\colon X \times X \to X$, given by direct sums of triangular diagrams. The observation that the rank of a direct sum of projective modules is the sum of the ranks will imply that m is compatible with filtrations; see Remark 3.14 below.

Returning briefly to the general case of an arbitrary (unfiltered) simplicial set X, let us justify the claim above that (32) is homotopic to the diagonal.

Proposition 3.8. *Suppose we are given maps $\Phi_X\colon |X| \to |X|$ for every simplicial set X that are natural in X, i.e., they assemble into a natural transformation Φ. Then Φ is naturally homotopic to the identity. Precisely, there is a homotopy*

$$|X| \times [0,1] \to |X|$$

from Φ_X to id_X for each X, and these homotopies are natural in X.

Proof. First, for $X = \Delta^n = \mathrm{Hom}_\Delta(-,[n])$ an n-simplex, one may take a straight line homotopy

$$|\Delta^n| \times [0,1] \to |\Delta^n|$$

from Φ_X to id_X, since Δ^n is a convex subset of $\mathbb{R}^n$. These straight line homotopies are natural in all morphisms of simplicial sets, in particular face maps and degeneracy maps. An arbitrary simplicial set is a colimit of simplices, so the result follows. □

Corollary 3.9. *For simplicial sets X, the geometric realization*

$$|X| \cong |\mathrm{es}(X)| \to |X| \times |X| \tag{34}$$

of the morphism in (31) *is naturally homotopic to the diagonal map.*

Proof. The natural transformation $\mathrm{es}(X) \Rightarrow X \times X$ is assembled from two natural transformations $\mathrm{es}(X) \to X$, with the property that the corresponding maps $|\mathrm{es}(X)| \to |X|$ are homotopic to the identity by Proposition 3.8. Therefore the map $|\mathrm{es}(X)| \to |X| \times |X|$ is naturally homotopic to the diagonal. □

Remark 3.10. Suppose as in Proposition 3.8 hold, we are given maps $\Phi_X\colon |X| \to |X|$ for every simplicial set X that are natural in X. Now if X is a *bisimplicial* set, for each $p \geq 0$, let X_p denote the simplicial set with $(X_p)_q = X_{p,q}$. Then we obtain a map $\Phi_X\colon |X| \to |X|$ by gluing the maps $\Phi_X\colon |X_p| \to |X_p|$ for each p. Then Proposition 3.8 implies that Φ_X is also naturally homotopic to the identity.

In particular, for a *bisimplicial* set X, the map $|X| \cong |\mathrm{es}(X)| \to |X| \times |X|$ is again homotopic to the diagonal, naturally in X.

Proposition 3.11. *Suppose*

$$F_0X \subset F_1X \subset \cdots \subset X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$$

is a filtered bisimplicial set. Define a filtration on the bisimplicial set $X \times X$ by

$$F_s(X \times X)_{p,q} = \bigcup_{u+v \leq s} (F_uX \times F_vX)_{p,q},$$

and similarly for $X \times X \times X$:

$$F_s(X \times X)_{p,q} = \bigcup_{t+u+v \leq s} (F_tX \times F_uX \times F_vX)_{p,q}.$$

Let $\Phi\colon \mathrm{es}(X) \Rightarrow X \times X$ be the natural transformation (31). *Suppose that Φ respects the filtrations, i.e. for each s, the natural transformation $\mathrm{es}(F_sX) \Rightarrow F_sX \times F_sX$ factors as*

$$\mathrm{es}(F_sX) \Rightarrow F_s(X \times X) \Rightarrow F_sX \times F_sX.$$

Then the diagram

$$\begin{tikzcd} |X| \arrow[r, "|\Phi|"] \arrow[d, "|\Phi|"'] & |X|\times|X| \arrow[d, "|\Phi|\times\mathrm{id}"] \\ |X|\times|X| \arrow[r, "\mathrm{id}\times|\Phi|"] & |X|\times|X|\times|X| \end{tikzcd}$$

commutes up to a homotopy

$$[0,1]\times|X|\to|X|\times|X|\times|X|.$$

Moreover, the homotopy may be chosen so that, for each $s\geq 0$, it restricts to a map

$$[0,1]\times|F_sX|\to|F_s(X\times X\times X)|.$$

Proof. We shall construct a map

$$G_X\colon [0,1]\times|X|\to|X|,$$

for every bisimplicial set X, which is natural in X. The map G_X will have the property that $G_X(0,-)\colon |X|\to|X|$ is the identity map, and, moreover, the composition

$$|X|\times[0,1]\xrightarrow{G_X}|X|\xrightarrow{(\mathrm{id}\times\Phi)\circ\Phi}|X|\times|X|\times|X| \tag{35}$$

is a natural homotopy from $(\mathrm{id}\times\Phi)\circ\Phi$ to $(\Phi\times\mathrm{id})\circ\Phi$.

Suppose we construct such maps G_X, natural in bisimplicial sets X. We now explain how this is enough to prove the Proposition. Suppose that X is a filtered bisimplicial set and suppose $\Phi\colon \mathrm{es}(X)\Rightarrow X\times X$ respects the filtration. Then for each $s\geq 0$, the map Φ restricts to a map

$$|F_s(X)|\to|F_s(X\times X)|,$$

and similarly, the map $(\mathrm{id}\times\Phi)\circ\Phi$, and also the map $(\Phi\times\mathrm{id})\circ\Phi$, restricts to a map

$$|F_s(X)|\to|F_s(X\times X\times X)|. \tag{36}$$

for each $s\geq 0$. Then applying (35) to $F_s(X)$ and combining with (36) yields a homotopy

$$|F_s(X)|\times[0,1]\longrightarrow|F_s(X\times X\times X)|$$

from $(\mathrm{id}\times\Phi)\circ\Phi$ to $(\Phi\times\mathrm{id})\circ\Phi$, as desired.

Now we construct the maps G_X. They will be defined first on the level of individual simplices, shown to be natural in face and degeneracy maps, and hence defined for all simplicial sets. Since they are natural in morphisms of simplicial sets, they then define maps on bisimplicial sets.

Consider Cartesian coordinates on the standard n-simplex for each n:

$$\Delta^n\cong\{(t_1,\ldots,t_n)\in\mathbb{R}^n : 0\leq t_1\leq t_2\leq\cdots\leq t_n\leq 1\}. \tag{37}$$

First let $g\colon [0,1]\to[0,1]$ be any nondecreasing continuous function with $g(0)=0$ and $g(1)=1$. Later on we will specialize g to a particular piecewise linear function. By applying g coordinatewise to (37), g defines self-maps on each simplex $g_n\colon \Delta^n\to\Delta^n$ given by

$$(t_1,\ldots,t_n)\mapsto(g(t_1),\ldots,g(t_n)).$$

For each n, the maps g_n and g_{n-1} are compatible with the $n+1$ face maps $\Delta^{n-1}\to\Delta^n$, which send $(t_1,\ldots,t_{n-1})$ to

$$(0,t_1,\ldots,t_{n-1}),\quad (t_1,\ldots,t_i,t_i,\ldots,t_{n-1})\text{ for } i=1,\ldots,n-1,\text{ and } (t_1,\ldots,t_{n-1},1)$$

respectively, as well as the n degeneracy maps $\Delta^n \to \Delta^{n-1}$ that send $(t_1, \ldots, t_n)$ to

$$(t_1, \ldots, \widehat{t_i}, \ldots, t_n)$$

for each $i = 1, \ldots, n$, respectively. Therefore, g determines maps $g_X \colon |X| \to |X|$ for any simplicial set X, naturally in X. If X is a bisimplicial set, then g also determines a map $|X_{p,\bullet}| \to |X_{p,\bullet}|$ for each p, and by gluing these, we get a map $g_X \colon |X| \to |X|$ as well, which is natural in morphisms of bisimplicial sets. Finally, let

$$G_X \colon [0,1] \times |X| \to |X| \tag{38}$$

be the straight line homotopy from the identity map to g_X; this is again natural in morphisms of bisimplicial sets.

Now we specialize to a particular g. Let $g \colon [0,1] \to [0,1]$ be the increasing, piecewise linear map that sends the intervals

$$\left[0, \tfrac{1}{4}\right], \quad \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \quad \left[\tfrac{1}{2}, 1\right]$$

linearly to the intervals

$$\left[0, \tfrac{1}{2}\right], \quad \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \quad \left[\tfrac{3}{4}, 1\right]$$

respectively. What remains only to show is that the composition

$$|X| \xrightarrow{g_X} |X| \xrightarrow{(\mathrm{id}\times\Phi)\circ\Phi} |X| \times |X| \times |X|$$

is equal to $(\Phi \times \mathrm{id}) \circ \Phi$. It suffices to perform this calculation in the case of a simplex

$$X = \Delta^n = \mathrm{Hom}_\Delta(-, [n]) \colon \Delta^{\mathrm{op}} \to \mathsf{Set}.$$

We first state the following lemma, without proof:

Lemma 3.12. *Let X be an n-simplex. In Cartesian coordinates, the map*

$$\Phi \colon |X| \to |X| \times |X|$$

is given by

$$(t_1, \ldots, t_n) \mapsto ((t'_1, \ldots, t'_i, 1, \ldots, 1), (0, \ldots, 0, t'_{i+1}, \ldots, t'_n))$$

for

$$0 \le t_1 \le \cdots \le t_i \le \tfrac{1}{2} \le t_{i+1} \le \cdots \le t_n \le 1$$

and where t'_ℓ denotes the result of applying to t_ℓ the appropriate linear rescaling

$$\left[0, \tfrac{1}{2}\right] \to [0,1] \quad \text{or} \quad \left[\tfrac{1}{2}, 1\right] \to [0,1].$$

Continue to let X be an n-simplex. Applying Lemma 3.12, in Cartesian coordinates, the map $(\mathrm{id} \times \Phi) \circ \Phi$ is given by

$$(t_1, \ldots, t_n) \mapsto ((t'_1, \ldots, t'_i, 1, \ldots, 1), (0, \ldots, 0, t'_{i+1}, \ldots, t'_j, 1, \ldots, 1), (0, \ldots, 0, t'_{j+1}, \ldots, t'_n))$$

for

$$0 \le t_1 \le \cdots \le t_i \le \tfrac{1}{2} \le t_{i+1} \le \cdots \le t_j \le \tfrac{3}{4} \le t_{j+1} \le \cdots \le t_n \le 1$$

and t'_ℓ denotes the result of applying to t_ℓ the appropriate linear rescaling

$$\left[0, \tfrac{1}{2}\right] \to [0,1], \quad \left[\tfrac{1}{2}, \tfrac{3}{4}\right] \to [0,1], \quad \text{or} \quad \left[\tfrac{3}{4}, 1\right] \to [0,1].$$

And the map $(\Phi \times \mathrm{id}) \circ \Phi$ has the same description as above, replacing the numbers $\frac{1}{2}$ and $\frac{3}{4}$ with $\frac{1}{4}$ and $\frac{1}{2}$ respectively. The equality

$$(\mathrm{id} \times \Phi) \circ \Phi \circ g_X = (\Phi \times \mathrm{id}) \circ \Phi$$

is now clear. This proves Proposition 3.11. □

For $X = N_\bullet S_\bullet(\mathrm{Proj}_\mathbb{Z})$ with $\mathrm{Proj}_\mathbb{Z}$ the Waldhausen category of finitely generated projective $\mathbb{Z}$-modules, $|X| = BK(\mathbb{Z})$ and we will use the filtered map $|X| \approx |\mathrm{es}(X)| \to |X| \times |X|$ to induce a *coproduct* on the Quillen spectral sequence. The *product* will be induced by a filtered map $|X| \times |X| \to |X|$ which is in some ways easier to comprehend, but depends on extra structure on the category $\mathrm{Proj}_\mathbb{Z}$, namely the symmetric monoidal structure given by direct sum of free $\mathbb{Z}$-modules. Strictly speaking, the direct sum operation $(P, P') \mapsto P \oplus P'$ is a choice, at least of the set underlying $P \oplus P'$—because of this, the definition of the product map $|X| \times |X| \to |X|$ looks a bit lengthy when spelled out, but hopefully will not be surprising.

Proposition 3.13. *Let*

$$X = N_\bullet S_\bullet(\mathrm{Proj}_\mathbb{Z}) \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$$

be the bisimplicial set with

$$X_{p,q} = N_q S_p(\mathrm{Proj}_\mathbb{Z})$$

as before. Define a product

$$m \colon X \times X \to X \tag{39}$$

given by maps

$$(X \times X)_{p,q} = X_{p,q} \times X_{p,q} \to X_{p,q}$$

which are direct sums of composable chains of morphisms of triangular diagrams. Precisely, we first choose for each pair of objects V, V' of $\mathrm{Proj}_\mathbb{Z}$ an object $m(V, V')$ and morphisms

$$V \to m(V, V') \leftarrow V'$$

satisfying the universal property of coproducts. In other words, $m(V, V')$ is a chosen model for the direct sum $V \oplus V'$, and there is a canonical associator and symmetry making $(\mathrm{Proj}_\mathbb{Z}, m)$ into a symmetric monoidal category. As usual, any two choices of m's will be canonically isomorphic, although the functions $N_0\mathrm{Proj}_\mathbb{Z} \times N_0\mathrm{Proj}_\mathbb{Z} \to N_0\mathrm{Proj}_\mathbb{Z}$ need not be equal. Relatedly, the product map $m : |X| \times |X| \to |X|$ that we will define depends on the choice of m, although its homotopy class as a filtered map will not. From now on we will fix such a choice and write simply $V \oplus V'$ for the chosen object $m(V, V')$.

Suppose now we are given two elements of $X_{p,q}$, given (after suppressing chosen cokernels from the notation, in other words suppressing from the notation all but the first row of each triangular diagram (13)*) by rectangular diagrams*

$$\begin{array}{ccccccc} 0 \longrightarrow & V_1^0 & \rightarrowtail \cdots \rightarrowtail & V_p^0 \,, & \quad 0 \longrightarrow & W_1^0 & \rightarrowtail \cdots \rightarrowtail & W_p^0 \\ & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ & \vdots & \cdots & \vdots & & \vdots & \cdots & \vdots \\ & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ 0 \longrightarrow & V_1^q & \rightarrowtail \cdots \rightarrowtail & V_p^q & \quad 0 \longrightarrow & W_1^q & \rightarrowtail \cdots \rightarrowtail & W_p^q. \end{array}$$

The notation that has been suppressed includes a choice of cokernel $V^k_{i,j}$ for each map $V^k_i \to V^k_j$. For ease of notation, we shall abbreviate rectangular diagrams of this form further, so that the diagrams above are abbreviated as

$$0 \to V_1^\bullet \to \cdots \to V_p^\bullet, \qquad 0 \to W_1^\bullet \to \cdots \to W_p^\bullet.$$

Then their product is defined to be the diagram

$$V_1^\bullet \oplus W_1^\bullet \to \cdots \to V_p^\bullet \oplus W_p^\bullet.$$

The chosen cokernels, which still do not appear in the notation, are the direct sums of the existing choices: namely, the chosen cokernel for

$$V_i^k \oplus W_i^k \to V_j^k \oplus W_j^k$$

is $V^k_{i,j} \oplus W^k_{i,j}$. Write $m\colon |X| \times |X| \to |X|$ and $\Phi\colon |X| \to |X| \times |X|$ also for the maps on spaces. Then we claim that product and coproduct are compatible:

$$\begin{tikzcd}
|X| \times |X| \arrow[rr, "m"] \arrow[d, "\Phi\times\Phi"] & & |X| \arrow[d, "\Phi"] \\
|X|\times|X|\times|X|\times|X| \arrow[rr, "(m\times m)\circ(\mathrm{id}\times s\times \mathrm{id})"] & & |X|\times|X|
\end{tikzcd}$$

where $s\colon X \times X \to X \times X$ exchanges first and second coordinate. The lower horizontal map is thus

$$(x_1, x_2, x_3, x_4) \mapsto (m(x_1, x_3), m(x_2, x_4)).$$

Proof. It suffices to show that the compositions

$$\mathrm{es}(X) \times \mathrm{es}(X) \xrightarrow{m} \mathrm{es}(X) \xrightarrow{\Phi} X \times X$$

and

$$\mathrm{es}(X) \times \mathrm{es}(X) \xrightarrow{\Phi\times\Phi} X \times X \times X \times X \xrightarrow{(m\times m)\circ(\mathrm{id}\times s\times \mathrm{id})} X \times X$$

are equal as morphisms of bisimplicial sets. This is direct from the definitions: both maps send a pair of (p, q)-simplices of $\mathrm{es}(X)$

$$((0 \to V_1^\bullet \to \cdots \to V_{2p+1}^\bullet), (0 \to W_1^\bullet \to \cdots \to W_{2p+1}^\bullet))$$

to

$$((0 \to V_1^\bullet \oplus W_1^\bullet \to \cdots \to V_{2p+1}^\bullet \oplus W_{2p+1}^\bullet), (0 \to V_{p+1,p+2}^\bullet \oplus W_{p+1,p+2}^\bullet \to \cdots \to V_{p+1,2p+1}^\bullet \oplus W_{p+1,2p+1}^\bullet),$$

and the proposition follows. ∎

Remark 3.14. We also record that the product m is evidently compatible with the filtration on X. Precisely, m restricts to a map

$$m\colon |F_sX| \times |F_tX| \to |F_{s+t}X| \tag{40}$$

for all $s, t \geq 0$.

Remark 3.15. We also need the following generalities on multiplicative spectral sequences.

Given a filtered space and a product that respects filtrations, we obtain (at this level of generality) a spectral sequence on relative singular homology with products on each page, and all differentials on each page are derivations of the product (Leibniz rule), and the isomorphism from $H_*(E_r) \to E_{r+1}$ is a multiplicative isomorphism with respect to the product on $H_*(E_r)$ induced from that on E_r.

Proposition 3.16. *Let*

$$X = N_\bullet S_\bullet(\mathrm{Proj}_{\mathbb{Z}})\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set},$$

with its rank filtration. Then the product $m\colon X \times X \to X$ is commutative up to homotopy, respecting the filtration. In other words, there is a homotopy

$$H\colon |X| \times |X| \times [0,1] \to |X|$$

from m to $m \circ s$, where $s\colon X \times X \to X \times X$ switches coordinates. Furthermore, H respects the filtration, in that it restricts to a map

$$H\colon |F_t X| \times |F_u X| \times [0,1] \to |F_{t+u} X|$$

for all $t, u \geq 0$.

Proof. We shall construct a morphism of bisimplicial sets

$$X \times X \times \Delta^{1,0} \to X$$

whose geometric realization is the desired H. Here

$$\Delta^{1,0} = \mathrm{Hom}_{\Delta\times\Delta}(-, ([1],[0])),$$

whose geometric realization is an interval $[0,1]$. For ease of notation, abbreviate

$$0 \to V_1^\bullet \to \cdots \to V_p^\bullet$$

for a (p,q)-simplex of X, as before. The notation stands for a chain of q morphisms of triangular diagrams of projective $\mathbb{Z}$-modules of size p.

For $i = 0, \ldots, p+1$, let $f_i \in \Delta([p],[1])$ be the morphism with $f_i(x) = 0$ if $x < i$ and $f_i(x) = 1$ if $x \geq i$. An element of $(X \times X \times \Delta^{1,0})_{p,q}$ is a triple

$$(0 \to V_1^\bullet \to \cdots \to V_p^\bullet, 0 \to W_1^\bullet \to \cdots \to W_p^\bullet, f_i)$$

for some $i \in \{0, \ldots p+1\}$. It is sent by H to

$$0 \to V_1^\bullet \oplus W_1^\bullet \to \cdots \to V_{i-1}^\bullet \oplus W_{i-1}^\bullet \to W_i^\bullet \oplus V_i^\bullet \to \cdots \to W_p^\bullet \oplus V_p^\bullet.$$

These maps

$$(X \times X \times \Delta^{1,0})_{p,q} \to X_{p,q}$$

are compatible with face and degeneracy maps, so we have produced a map of bisimplicial sets, which restricts to m and $m \circ s$ at $t = 0$ and $t = 1$, respectively. Finally, H obviously respects filtration: it restricts to a map of bisimplicial sets, for any $t, u \geq 0$,

$$F_t X \times F_u X \times \Delta^{1,0} \to F_{t+u} X,$$

since the rank of a direct sum of modules of rank t and u is $t + u$. $\square$

3.2. **Monoidality for the spectral sequence of a filtered space.** We recall a certain monoidality of the spectral sequence associated to a filtered space. This seems well known in the algebraic topology literature, so we list the precise statements we need and give some references: [CE99, Dou59a, Dou59b]. See also [Rog, Chapter 6] and [Goe].

3.2.1. *Filtered spaces.* In this section we write H_* for singular homology with coefficients in some field. Recall that for a topological space X filtered by subcomplexes $F_tX \subset X$, there is a spectral sequence

$$E^1_{s,t} = E^1_{s,t}(X) = H_{s+t}(F_tX, F_{t-1}X) \Rightarrow H_*(X),$$

with convergence assuming the filtration is bounded below, for instance $F_{-1}X = \emptyset$, and exhaustive: $\mathrm{colim}_t\, H_*(F_tX) \to H_*(X)$ is an isomorphism. For definiteness, let us work with the construction of this spectral sequence given in [CE99, Chap. XV, §7], see especially Example 3 on page 335. This agrees with the construction in [Dou59a, Section II.C] apart from notation (in particular, the latter writes $\pi(p,q)$ for what the former denotes $H(p,q)$). This spectral sequence is natural with respect to all maps of filtered spaces: that is, continuous maps $X \to X'$ sending F_tX into F_tX'.

3.2.2. *Products of filtered spaces.* If X' and X'' are filtered spaces, then the Cartesian product $X = X' \times X''$ inherits a filtration, namely

$$F_tX = \mathrm{Im}\Big(\coprod_{u+v\le t} F_uX' \times F_vX'' \to X' \times X'' = X\Big).$$

In this situation the inclusion $F_uX' \times F_vX'' \hookrightarrow F_{u+v}X$ induces a chain map

$$C_*(F_uX') \otimes C_*(F_vX'') \to C_*(F_{u+v}X)$$

defined by the chain-level cross product. This inclusion sends both subspaces $F_uX' \times F_{v-1}X''$ and $F_{u-1}X' \times F_vX''$ into $F_{u+v-1}X$, so the cross product factors over a chain map

$$C_*(F_uX', F_{u-1}X') \otimes C_*(F_vX'', F_{v-1}X'') \to C_*(F_{u+v}X, F_{u+v-1}X). \tag{41}$$

Passing to homology then gives a homomorphism

$$\phi^1 : E^1_{p,q}(X') \otimes E^1_{p',q'}(X'') \to E^1_{p+p',q+q'}(X) \tag{42}$$

which we will call the *exterior product*. In brief, these make $X \mapsto E^1_{*,*}(X)$ into a lax monoidal functor from filtered spaces to bigraded vector spaces.

Similarly, filtering $H_*(X)$ by the images of the $H_*(F_tX)$, and similarly for $H_*(X')$ and $H_*(X'')$, the cross product $H_*(X') \otimes H_*(X'') \to H_*(X)$ descends to a pairing

$$\mathrm{Gr}_p H_{p+q}(X') \otimes \mathrm{Gr}_{p'} H_{p'+q'}(X'') \to \mathrm{Gr}_{p+p'} H_{p+p'+q+q'}(X). \tag{43}$$

Proposition 3.17. *In the above setting there are pairings*

$$\phi^r : E^r_{p,q}(X') \otimes E^r_{p',q'}(X'') \to E^r_{p+p',q+q'}(X)$$

for all $r \ge 1$, agreeing with (42) for $r=1$, satisfying the Leibniz rule

$$d^r\phi^r(x' \otimes x'') = \phi^r((d^rx') \otimes x'' + (-1)^{p+q}x' \otimes (d^rx''))$$

and such that the induced pairing on homology of rth pages is identified with ϕ^{r+1}. (Such pairings are of course unique if they exist since each determines the next—the main content is the Leibniz rule so that ϕ^r descends to a pairing on homology *of rth pages.)*

Moreover, the induced pairing ϕ^∞ of E^∞ pages is identified with (43). Finally, if X' and X'' are CW complexes filtered by subcomplexes, then the homomorphisms

$$\phi^r : E^r_{*,*}(X') \otimes E^r_{*,*}(X'') \to E^r_{*,*}(X),$$

obtained by taking direct sum over all p, q, p', q', are isomorphisms for all $r \geq 1$.

Proof sketch. This is mostly contained in [Dou59b, Théorème II.A]. In the notation from there, we should set

$$\pi(q,r) = \bigoplus_n H_n(F_{-q}X, F_{-r}X)$$

for $-\infty \leq q \leq r \leq \infty$ (where we write $F_{-\infty}X = \emptyset$ and $F_\infty X = X$), let

$$\eta\colon \pi(q,r) \to \pi(q',r')$$

be defined by functoriality of homology for $q' \leq q$ and $r' \leq r$, and let

$$\partial\colon \pi(q,r) \to \pi(r,s)$$

be the connecting homomorphism for the triple (F_qX, F_rX, F_sX) for $-\infty \leq q \leq r \leq s \leq \infty$. This data is called a *système spectraux* in [Dou59a] and a *Cartan–Eilenberg system* in many other places, and satisfies the axioms (SP.1)–(SP.5) of [CE99].

Defining *systèmes spectraux* π' and π'' associated to the filtered spaces X' and X'' in the same manner, the inclusions $F_{-n}X' \times F_{-q}X'' \hookrightarrow F_{-n-q}X$ then induce homomorphisms

$$\pi'(n, n+r) \otimes \pi''(q, q+r) \xrightarrow{\phi_r} \pi(n+q, n+q+r) = H_*(F_{-n-q}X, F_{-n-q-r}X)$$

for all $r \geq 1$, in the same way as (42) which is the special case $r = 1$. These homomorphisms satisfy the assumptions of Douady's theorem, which then gives the stated result, apart from the claim that the ϕ^r define isomorphisms after taking direct sum over all bidegrees. The original reference [Dou59b] in fact omits the proof, but details can be found elsewhere, for instance [Goe] or [Rog]. The verification of axioms (SPP.1) and (SPP.2) in Douady's theorem is as in [Rog, Proposition 6.3.12].

It remains to see that the exterior products ϕ^r become isomorphisms after taking direct sum over all bidegrees, when X' and X'' are CW complexes filtered by subcomplexes. In that case we can identify relative homology with reduced homology of the quotient space: $E^1_{p,q}(X') = \widetilde{H}_{p+q}(F_pX'/F_{p-1}X')$, and similarly for X'' and X. Then ϕ^1 is induced by maps $(F_uX'/F_{u-1}X') \wedge (F_vX''/F_{v-1}X'') \to F_{u+v}X/F_{u+v-1}X$ arising from the inclusions $F_uX' \times F_vX'' \hookrightarrow F_{u+v}X$. Taking wedge sum over all filtrations, we see that ϕ^1 is induced from a single map of pointed spaces

$$\Big(\bigvee_{s\in\mathbb{Z}} F_sX'/F_{s-1}X'\Big) \wedge \Big(\bigvee_{s\in\mathbb{Z}} F_sX''/F_{s-1}X''\Big) \longrightarrow \bigvee_{s\in\mathbb{Z}} F_sX/F_{s-1}X,$$

and the claim about isomorphism follows for $r = 1$ from the Künneth theorem and the observation that this map is in fact a homeomorphism when $X = X' \times X''$ is given the CW topology. But then inductively ϕ^r is also an isomorphism for higher r, again by the Künneth theorem. $\square$

3.3. **Hopf algebra structure on the Quillen spectral sequence.** The results earlier in this section combine to yield the following extra structures on ${}^{Q}E$, the homological Quillen spectral sequence.

Theorem 3.18.

(1) *The homological Quillen spectral sequence*
$$ {}^{Q}E^1_{s,t} = H_t(\mathrm{GL}_s(\mathbb{Z}), \mathrm{St}_s \otimes \mathbb{Q}) \Rightarrow H_*(BK(\mathbb{Z})) $$
is a spectral sequence of Hopf algebras. That is, for each $r \geq 0$, there are maps
$$ m^r \colon E^r_{s,t} \otimes E^r_{s',t'} \to E^r_{s+s',t+t'}, \qquad \Delta^r \colon E^r_{s,t} \to \bigoplus_{\substack{s'+s''=s \\ t'+t''=t}} E^r_{s',t'} \otimes E^r_{s'',t''}, $$
induced from
$$ m\colon BK(\mathbb{Z}) \times BK(\mathbb{Z}) \to BK(\mathbb{Z}) \quad \text{and} \quad \Phi\colon BK(\mathbb{Z}) \to BK(\mathbb{Z}) \times BK(\mathbb{Z}) $$
respectively, making E^r a bigraded Hopf algebra. Moreover, the differential
$$ d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1} $$
is compatible with product and coproduct in the sense that
$$ d^r(x_1 \cdot x_2) = d^r(x_1) \cdot x_2 + (-1)^{p_1+q_1} x \cdot d^r(x_2) $$
for $x_i \in E^r_{p_i,q_i}$, where $x \cdot y$ denotes the product m^r; and
$$ \Delta^r \circ d^r = (d^r \otimes 1 \pm 1 \otimes d^r) \circ \Delta^r. $$
More precisely, the sign "$\pm$" can be expressed in terms of the symmetry
$$ \begin{aligned} T : E^r_{p,q} \otimes E^r_{p',q'} &\xrightarrow{\cong} E^r_{p',q'} \otimes E^r_{p,q} \\ x \otimes y \quad &\mapsto (-1)^{(p+q)(p'+q')} y \otimes x; \end{aligned} $$
the Leibniz rule for the coproduct should then read $\Delta^r \circ d^r = (d^r \otimes 1 + T \circ (d^r \otimes 1) \circ T) \circ \Delta^r$.

(2) *There are isomorphisms $\mathbb{Q} \to E^r_{0,0} \to \mathbb{Q}$ acting as unit and counit respectively, making each page E^r into a bigraded Hopf algebra.*

(3) *Giving the homology of the rth page the Hopf algebra structure induced by m^r and Δ^r, the isomorphism $E^{r+1} = H(E^r, d^r)$ is a Hopf algebra isomorphism.*

(4) *The filtration of $H_*(BK(\mathbb{Z}); \mathbb{Q})$ induced by convergence of the spectral sequence is multiplicative and comultiplicative, where the product on $H_*(BK(\mathbb{Z}); \mathbb{Q})$ is induced by $m : BK(\mathbb{Z}) \times BK(\mathbb{Z}) \to BK(\mathbb{Z})$ (in turn constructed from the $\oplus$ operation on the Waldhausen construction; this also agrees with the product induced by the loop space structure) and the coproduct on $H_*(BK(\mathbb{Z}); \mathbb{Q})$ is induced by the diagonal map (dual to cup product). With respect to the induced Hopf algebra structure on $\mathrm{Gr} H_*(BK(\mathbb{Z}); \mathbb{Q})$, the isomorphisms $E^\infty_{p,q} = \mathrm{Gr}_p H_{p+q}(BK(\mathbb{Z}); \mathbb{Q})$ form an isomorphism of bigraded Hopf algebras.*

(5) *The product on E^r is graded-commutative for all $r \geq 1$:*
$$ x_1 \cdot x_2 = (-1)^{(p_1+q_1)(p_2+q_2)} x_2 \cdot x_1, $$
for x_i in $E^r_{p_i,q_i}$.

(6) The coproduct Δ^r is graded co-commutative for $r = \infty$ but not necessarily for finite r. (We are of course asserting that it is co-associative for all r, which is part of the axioms for Hopf algebras.)

Proof. Setting $X' = X'' = BK(\mathbb{Z})$ in Proposition 3.17, we obtain isomorphisms

$$E^r_{*,*}(BK(\mathbb{Z})) \otimes E^r_{*,*}(BK(\mathbb{Z})) \xrightarrow{\phi^r} E^r_{*,*}(BK(\mathbb{Z}) \times BK(\mathbb{Z})) \tag{44}$$

satisfying the Leibniz rule on each page, as well as compatibility between pages and with the abutments.

Now we combine with the space-level products and coproducts and functoriality of the spectral sequence with respect to filtered maps, using Corollary 3.9, Propositions 3.11, 3.13, 3.16, and Remark 3.14. For instance, the space-level coproduct $\Phi : BK(\mathbb{Z}) \to BK(\mathbb{Z}) \times BK(\mathbb{Z})$ is a map of filtered spaces and hence induces a map of spectral sequences

$$E^r_{*,*}(BK(\mathbb{Z})) \xrightarrow{\Phi_*} E^r_{*,*}(BK(\mathbb{Z}) \times BK(\mathbb{Z}))$$

which we combine with the inverse of (44) to obtain a coproduct on rth pages of the Quillen spectral sequence. Similarly for product, while the unit and counit come from space-level maps $\{\text{point}\} \to BK(\mathbb{Z}) \to \{\text{point}\}$, where the one-point space is filtered as $F_{-1} = \emptyset \subset \{\text{point}\} = F_0$.

To see that the coproduct on the spectral sequence is coassociative, we write $I = [0, 1]$ filtered as $\emptyset = F_{-1}I \subset F_0 I = I$ and first observe that the two injections $BK(\mathbb{Z}) \hookrightarrow I \times BK(\mathbb{Z})$, given by $x \mapsto (0, x)$ and $x \mapsto (1, x)$, induce equal maps of spectral sequences, since both are one-sided inverses to the isomorphism of spectral sequences induced by the projection $I \times BK(\mathbb{Z}) \to BK(\mathbb{Z})$. Coassociativity then follows from the space-level homotopy

$$I \times BK(\mathbb{Z}) \to BK(\mathbb{Z}) \times BK(\mathbb{Z}) \times BK(\mathbb{Z}),$$

observing that this is in fact a filtered map.

To see that the coproduct is co-commutative on the E^∞ page, we use that the space-level map Φ is homotopic to the diagonal map. Therefore the induced coproduct $\Phi_* : H_*(BK(\mathbb{Z})) \to H_*(BK(\mathbb{Z})) \otimes H_*(BK(\mathbb{Z}))$ is co-commutative, but then this also holds for the induced map of associated gradeds, which is identified with $\phi^\infty : E^\infty_{*,*} \to E^\infty_{*,*} \otimes E^\infty_{*,*}$.

All other properties follow from the corresponding space-level properties in a similar way. ☐

4. Proof of Theorems 1.1 and 1.2

4.1. **A product on tropical moduli spaces.** Towards a proof of Theorem 1.1, we shall first study a graded-commutative product on $W_0 H^*_c(\mathcal{A})$, which can be interpreted both in terms of products of abelian varieties and of tropical abelian varieties. As in the proof of Proposition 2.8, we have short exact sequences

$$0 \to H^{\mathrm{BM}}_k(A^{\mathrm{trop}}_g) \xrightarrow{\iota} H^{\mathrm{BM}}_k(A^{\mathrm{trop}}_g, A^{\mathrm{trop}}_{g-1}) \xrightarrow{\partial} H^{\mathrm{BM}}_{k-1}(A^{\mathrm{trop}}_{g-1}) \to 0 \tag{45}$$

for all k and g. These follow from the fact that $H^{\mathrm{BM}}_*(A^{\mathrm{trop}}_{g-1}) \to H^{\mathrm{BM}}_*(A^{\mathrm{trop}}_g)$ is zero, established in (11).

Now let $(P^{(g)}[-1], d)$ denote the degree-shifted *perfect cone complex*, whose definition and properties we shall now recall from [BBC+24]. The complex $P^{(g)}[-1]$ is a rational chain complex with differential of degree -1, with generators $[\sigma, \omega]$ in degree $\dim(\sigma)$, where σ is a perfect cone and ω is an orientation of the linear span of σ. Relations are given by $[\sigma, \omega] = \pm[\sigma', \omega']$ if σ and

σ' are in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit, where the sign in the relation depends on whether ω and ω' agree or disagree according to the induced action of $\mathrm{GL}_g(\mathbb{Z})$ on orientation. The boundary of $[\sigma, \omega]$ is a sum of codimension 1 faces of σ with induced orientation. Then from [BBC$^+$24] we have $H_k^{\mathrm{BM}}(A_g^{\mathrm{trop}}) \cong H_k(P^{(g)}[-1])$. Moreover, there are natural inclusions $P^{(g-1)} \to P^{(g)}$ such that

$$\begin{tikzcd}
H_k(P^{(g-1)}[-1]) \arrow[r] \arrow[d, "\cong"'] & H_k(P^{(g)}[-1]) \arrow[d, "\cong"] \\
H_k^{\mathrm{BM}}(A_{g-1}^{\mathrm{trop}}) \arrow[r] & H_k^{\mathrm{BM}}(A_g^{\mathrm{trop}})
\end{tikzcd}$$

commutes. Therefore from (45) we obtain short exact sequences

$$0 \to H_k(P^{(g)}[-1]) \xrightarrow{\iota} H_k((P^{(g)}/P^{(g-1)})[-1]) \xrightarrow{\partial} H_{k-1}(P^{(g-1)}[-1]) \to 0$$

for all k and g.

Now we construct a product map and prove a Leibniz rule. Consider the natural continuous maps

$$A_g^{\mathrm{trop}} \times A_h^{\mathrm{trop}} \to A_{g+h}^{\mathrm{trop}} \tag{46}$$

induced by block sum of positive semidefinite forms. These maps extend to

$$(A_g^{\mathrm{trop}} \cup \{\infty\}, \infty) \times (A_h^{\mathrm{trop}} \cup \{\infty\}, \infty) \to (A_{g+h}^{\mathrm{trop}} \cup \{\infty\}, \infty) \tag{47}$$

the one-point compactifications with added point ∞. Hence we obtain

$$H_k^{\mathrm{BM}}(A_g^{\mathrm{trop}}) \otimes H_\ell^{\mathrm{BM}}(A_h^{\mathrm{trop}}) \to H_{k+\ell}^{\mathrm{BM}}(A_{g+h}^{\mathrm{trop}}) \tag{48}$$

Similarly, we have

$$H_k^{\mathrm{BM}}(A_g^{\mathrm{trop}}, A_{g-1}^{\mathrm{trop}}) \otimes H_\ell^{\mathrm{BM}}(A_h^{\mathrm{trop}}, A_{h-1}^{\mathrm{trop}}) \to H_{k+\ell}^{\mathrm{BM}}(A_{g+h}^{\mathrm{trop}}, A_{g+h-1}^{\mathrm{trop}}) \tag{49}$$

Both (48) and (49) are induced from

$$m\colon P^{(g)}[-1] \otimes P^{(g')}[-1] \to P^{(g+g')}[-1], \quad [\sigma, \omega] \otimes [\sigma', \omega'] \mapsto [\sigma \times \sigma', \omega \wedge \omega'] \tag{50}$$

where, for cones $\sigma \subset \mathrm{Sym}^2((\mathbb{R}^g)^\vee)$ and $\sigma' \subset \mathrm{Sym}^2((\mathbb{R}^{g'})^\vee)$, we take $\sigma \times \sigma' \in \mathrm{Sym}^2((\mathbb{R}^{g'})^\vee) \times \mathrm{Sym}^2((\mathbb{R}^{g})^\vee) \subset \mathrm{Sym}^2((\mathbb{R}^{g+g'})^\vee)$ as the cone of block sums of symmetric bilinear forms in σ and σ'. The product $m(\sigma, \sigma')$ shall be denoted by $\sigma.\sigma'$ henceforth, and similarly for the products (48) and (49). Note that m is graded-commutative.

Consider the bigraded vector space

$$\mathcal{A}^{BM} = \bigoplus_n \mathcal{A}_n^{BM} \quad \text{where} \quad \mathcal{A}_n^{BM} = \bigoplus_{s \le n} H_n^{BM}(A_s^{\mathrm{trop}}; \mathbb{Q}),$$

where $H_n^{\mathrm{BM}}(A_s^{\mathrm{trop}}; \mathbb{Q})$ is in bidegree $(s, n-s)$. It is the graded dual of $W_0 H_c^*(\mathcal{A})$. Then we have the following conclusion.

Proposition 4.1. *The maps* (48) *equip* $\mathcal{A}^{\mathrm{BM}}$ *with the structure of a* $\mathbb{Q}$*-algebra, which is graded-commutative with respect to its total grading.*

Proof. The graded-commutativity of the product on $\mathcal{A}^{\mathrm{BM}}$ follows from graded-commutativity of the product (50) on perfect cone complexes. $\square$

Remark 4.2. The algebraic moduli space $\mathcal{A}$ has a natural space-level commutative product, given on connected components by

$$\mathcal{A}_g \times \mathcal{A}_h \to \mathcal{A}_{g+h}; \quad (A_1, A_2) \mapsto A_1 \times A_2. \tag{51}$$

This product morphism $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is proper, and the induced pullback map

$$W_0 H_c^*(\mathcal{A}) \to W_0 H_c^*(\mathcal{A}) \otimes W_0 H_c^*(\mathcal{A})$$

agrees with (48) via the standard comparison isomorphisms and dualities.

To see this agreement, one may first note that the tropicalization of the algebraic product (51) is the tropical product defined in (46), induced by block sum of positive semidefinite quadratic forms. Then, use the fact that the Berkovich analytic skeleton of a product of abelian varieties over a valued field is the product of the skeletons, as principally polarized tropical abelian varieties. This is because the skeleton can be read off via non-archimedean analytic uniformization from the Raynaud cross diagram, as in [FRSS18, Section 3.2], and the Raynaud cross of the product is the product of the Raynaud crosses of the factors.

We also note that the algebraic product (51) extends to additive families of toroidal compactifications, such as the perfect cone compactifications. The fact that (46) is the tropicalization of (51) is well-known in this context. See [GHT18, Proposition 9].

The following version of the Leibniz rule will be used in the next subsection.

Proposition 4.3. *Let $\alpha \in H_k^{\mathrm{BM}}(A_g^{\mathrm{trop}}, A_{g-1}^{\mathrm{trop}})$ and $\beta \in H_\ell^{\mathrm{BM}}(A_h^{\mathrm{trop}}, A_{h-1}^{\mathrm{trop}})$. Then, with ι and ∂ as in* (45)*, we have*

$$\iota\partial(\alpha.\beta) = (\iota\partial\alpha).\beta + (-1)^{\deg(\alpha)}\alpha.(\iota\partial\beta).$$

Furthermore, the map ι is a homomorphism for the multiplication maps (48) *and* (49)*.*

Proof. The key point is that the product (50) satisfies the graded Leibniz rule. More precisely, (50) is a morphism of chain complexes:

$$d([\sigma,\omega].[\sigma',\omega']) = d[\sigma,\omega].[\sigma',\omega'] + (-1)^{\dim(\sigma)}[\sigma,\omega].d[\sigma',\omega']$$

by properties of faces of products of polyhedral cones. Then the Proposition follows since the product (49) is induced from this product map on perfect cone complexes. ☐

4.2. **Hopf structure on $W_0 H_c^*(\mathcal{A})$.** Let ${}^Q E^1$ denote the E^1 page of the homological Quillen spectral sequence. It is a graded-commutative bigraded Hopf algebra by Theorem 3.18. The graded-commutativity is with respect to the total degree.

The vector space ${}^Q E^1_{1,0}$ is 1-dimensional and plays an important role in the Hopf structure on ${}^Q E^1$, so we pause briefly to establish a canonical choice of generator. The fundamental group $\pi_1(BK(\mathbb{Z})) = K_0(\mathbb{Z}) \cong \mathbb{Z}$ is infinite cyclic, and is in fact generated by the loop corresponding to any 1-simplex $(0 \subset P_{0,1}) \in N_0 S_1(\mathrm{Proj}_{\mathbb{Z}})$ for $P_{0,1}$ of rank 1 (that is, $P_{0,1}$ is isomorphic as an abelian group to $\mathbb{Z}$). Any two choices of $P_{0,1}$ will be isomorphic, and lead to homotopic loops in the based space $BK(\mathbb{Z})$. Let us write

$$e \colon S^1 \to BK(\mathbb{Z})$$

for any based loop in this homotopy class. The loop e can also be interpreted as the generator of the fundamental group of $\mathrm{gr}_1 BK(\mathbb{Z}) = (F_1 BK(\mathbb{Z}))/(F_0 BK(\mathbb{Z}))$. Hence we obtain a canonical generator

$$e_*([S^1]) \in H_1(F_1BK(\mathbb{Z}), F_0BK(\mathbb{Z}); \mathbb{Q}) = {}^{Q}E^1_{1,0} \cong \mathbb{Q},$$

where $[S^1]$ denotes the fundamental class of the circle $S^1 = \Delta^1/(\partial\Delta^1)$. Let us henceforth use the same notation $e \in {}^{Q}E^1_{1,0}$ for this homology class, as well as for its image $e \in {}^{Q}E^r_{1,0}$ for all subsequent pages $r \leq \infty$.

Proposition 4.4. *With $e \in {}^{Q}E^1_{1,0}$ as above, there is a canonical isomorphism of graded-commutative algebras*

$$\mathcal{A}^{BM} \otimes_{\mathbb{Q}} \mathbb{Q}[\epsilon]/\epsilon^2 \xrightarrow{\sim} {}^{Q}E^1 \ ,$$

sending ϵ to e, where $\mathbb{Q}[\epsilon]/\epsilon^2 \cong \bigwedge \mathbb{Q}\epsilon$ is the graded exterior algebra generated by ϵ.

Proof. Let $e \in {}^{Q}E^1_{1,0} \cong \mathbb{Q}$ be the generator chosen above. (Any non-zero multiple thereof would work equally well for the following argument). Since the coproduct Δ on ${}^{Q}E^1$ respects the bigrading, we must have

$$\Delta\, e = 1 \otimes e + e \otimes 1 \ .$$

By graded-commutativity, or using the fact that ${}^{Q}E^1_{2,0}$ vanishes, we have $e^2 = 0$.

Recall from (45) the short exact sequences

$$0 \to H^{\mathrm{BM}}_{s+t}(A^{\mathrm{trop}}_s) \xrightarrow{\iota} H^{\mathrm{BM}}_{s+t}(A^{\mathrm{trop}}_s, A^{\mathrm{trop}}_{s-1}) \xrightarrow{\partial} H^{\mathrm{BM}}_{s+t-1}(A^{\mathrm{trop}}_{s-1}) \to 0, \tag{52}$$

where the middle term is isomorphic to ${}^{Q}E^1_{s,t}$. The multiplication on ${}^{Q}E^1$ is induced by block sum of matrices, as in Proposition 2.15, and hence coincides with (49). Therefore, for any element $\alpha \in H^{BM}_{s+t-1}(A^{\mathrm{trop}}_{s-1}; \mathbb{Q})$, Proposition 4.3 implies the following formula in ${}^{Q}E^1$:

$$\iota\partial(e.\iota\alpha) = \iota\partial(e).\iota\alpha \tag{53}$$

since the second term in the Leibniz rule is $e.(\iota\partial\iota\alpha)$ which vanishes by (52). For $(s,t) = (1,0)$, the sequence (52) reduces to an isomorphism

$${}^{Q}E^1_{1,0} \overset{\partial}{\cong} H^{BM}_0(A^{\mathrm{trop}}_0; \mathbb{Q}) \ ,$$

and hence by injectivity of ι, we can write $\iota\partial(e) = \lambda 1 \in {}^{Q}E^1_{0,0}$ for $\lambda \in \mathbb{Q}^\times$ and $1 \in {}^{Q}E^1_{0,1}$ the unit element in the algebra structure. (In fact it can be seen that $\lambda = 1$, but we shall not need that right now.) Then $\partial(e.\iota\alpha) = \lambda\alpha$, so that $\alpha \mapsto \lambda^{-1}e.\iota\alpha$ is a right-inverse to ∂, splitting the short exact sequence (52).

Let $\mathcal{A}^{\mathrm{BM}}[-1]$ denote the shift by $(-1,0)$ in bidegree, so $\mathcal{A}^{\mathrm{BM}}[-1]_{s,t} = \mathcal{A}^{\mathrm{BM}}_{s-1,t}$. We now define the following diagram, which, we shall then argue, is commutative.

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{A}^{BM} & \longrightarrow & \mathcal{A}^{BM} \oplus \mathcal{A}^{BM}[-1] & \longrightarrow & \mathcal{A}^{BM}[-1] \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathcal{A}^{BM} & \longrightarrow & {}^{Q}E^1_{*,*} & \xrightarrow{\partial} & \mathcal{A}^{BM}[-1] \longrightarrow 0
\end{array}$$

Here the top row is the canonical split short exact sequence, and the bottom row is (52) in bidegree (s,t). The vertical map in the middle is

$$(\alpha, \beta) \mapsto \iota\alpha + \lambda^{-1}e.\iota\beta$$

and the left and right vertical maps are the identity. By (53), $\partial(\iota\alpha+\lambda^{-1}e.\iota\beta)=\beta$ and therefore the diagram commutes, and all vertical maps must be isomorphisms. In particular, we deduce a canonical isomorphism $\mathcal{A}^{BM}\otimes_{\mathbb{Q}}\mathbb{Q}[\epsilon]/\epsilon^2\cong {}^{Q}E^1$ on the level of bigraded vector spaces.

Finally, the natural morphism $\mathcal{A}^{\mathrm{BM}}\to {}^{Q}E^1$ arising from the above isomorphism is a morphism of graded-commutative algebras, i.e., respects the products. Indeed, the products on domain and codomain are (48) and (49), respectively, and are both induced from the same product map (50) on perfect complexes. It follows from this, together with the definition of the middle vertical map in the above diagram, that $\mathcal{A}^{BM}\otimes_{\mathbb{Q}}\mathbb{Q}[\epsilon]/\epsilon^2\cong {}^{Q}E^1$ is an isomorphism of algebras. ☐

Proof of Theorem 1.1. For any generator $e\in {}^{Q}E^1_{1,0}\cong\mathbb{Q}$, we must have

$$\Delta(e)=1\otimes e+e\otimes 1$$

since the coproduct Δ on ${}^{Q}E^1$ respects the bigrading. Therefore the ideal generated by e is also a co-ideal, and the quotient Hopf algebra ${}^{Q}E^1/(e)$ is isomorphic with $\mathcal{A}^{\mathrm{BM}}$ by Proposition 4.4. This yields a Hopf structure on $\mathcal{A}^{\mathrm{BM}}$, and dualizing yields a Hopf structure on $W_0H_c^*(\mathcal{A})$. ☐

Next, we show that there is a natural injection from the bigraded vector space $\Omega_c^*[-1]$ into the subspace of primitives $\mathrm{Prim}(W_0H_c^*(\mathcal{A}))$. We recall the canonical differential forms for GL_g and their basic properties from §2.4.

Proof of Theorem 1.2. By [Bro23] there is an injective map of graded $\mathbb{R}$-vector spaces

$$\Omega_c^*(g)[-1]\otimes\mathbb{R}\to H_c^*(P_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R})\subset {}^{Q}E_1^{g,*}\otimes\mathbb{R} \tag{54}$$

for all $g>1$ odd (see Section 2.4 for further detail).

It remains to show that the image of $\Omega_c^*(g)[-1]$ in ${}^{Q}E_1$ consists of primitive elements. Let $\omega\in\Omega_c^*(g)$ be homogeneous of compact type where $g>1$ is odd. We may assume that it is of the form

$$\omega^I=\omega^{4i_1+1}\wedge\omega^{4i_2+1}\wedge\ldots\wedge\omega^{4i_k+1}$$

where $I=\{i_1,\ldots,i_k\}$ are distinct and $4i_k+1=2g-1$. From (20) we deduce that

$$\omega^I_{X\oplus Y}=\sum_{I=J\cup K}\omega^J_X\otimes\omega^K_Y$$

where the sum is over all decompositions of I into a disjoint union of (possibly empty) sets J,K. Suppose that X and Y both have positive rank. Since one of ω^J and ω^K must necessarily have ω^{2g-1} as a factor, it follows from (21) that $\omega^J_X\otimes\omega^K_Y=0$ for all J,K and we deduce that $\omega^I_{X\oplus Y}$ is identically zero. If $\Delta'=\sum_{m,n\geq 1}\Delta_{m,n}$ denotes the reduced coproduct, defined by $\Delta=\mathrm{id}\otimes 1+1\otimes\mathrm{id}+\Delta'$, then we have shown that $\Delta'\omega=0$ for all $\omega\in\Omega_c(g)$ since the restriction of ω to the image of $\mathcal{A}_m^{\mathrm{trop}}\times\mathcal{A}_n^{\mathrm{trop}}$ for $m+n=g$ and $m,n>0$ is zero. It follows that the image of $\Omega_c(g)[-1]$ in ${}^{Q}E_1$ is primitive. The theorem follows by quotienting by ϵ. ☐

4.3. **Consequences of Theorem 1.2 and a conjecture.** Theorem 1.2 immediately implies:

Theorem 4.5. *There is a canonical morphism of bigraded Hopf algebras*

$$T(\Omega_c^*[-1])\otimes\mathbb{R}\longrightarrow W_0H_c^*(\mathcal{A};\mathbb{R}) \tag{55}$$

which is injective on $\Omega_c^*[-1]$.

The map (55) induces a map of graded Lie algebras

$$\mathbb{L}(\Omega_c^*[-1]) \longrightarrow \mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{R})) \ , \tag{56}$$

where $\mathbb{L}$ denotes the free (graded-commutative) Lie algebra on $\Omega_c^*[-1]$.

Conjecture 4.6. *The map* (55) *is injective. Equivalently, it gives an injective map from the free Lie algebra on* $\Omega_c^*[-1]$ *into* $\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{R}))$.

The later sections of this paper provide various kinds of evidence for this conjecture.

Remark 4.7. A similar conjecture in [Bro21b] states that the free Lie algebra on a space isomorphic to $\Omega_c^*[-1]$ injects into the cohomology of the commutative graph complex (or equivalently, in the cohomology of the moduli spaces of tropical curves). There are some key differences between these two conjectures: the map in Conjecture 4.6 is given explicitly and respects the bigrading; the one in [Bro21b] is not explicit and does not respect the grading by genus in general. These two conjectures are not related by the tropical Torelli map: one can show that tropical Torelli map is zero on the image of almost all canonical forms which lie above the 'diagonal' where degree equals twice the genus [Bro23, §14.6]. It is, however, expected to be an isomorphism on the Lie subalgebra generated by the ω^{4k+1} which lies on the diagonal.

5. Graph spectral sequences

Having produced an injection from the vector space $\Omega_c^*[-1]$ into the primitives of $W_0H_c^*(\mathcal{A})$, we now work toward proving that the images of the elements $\omega^5, \omega^9, \ldots, \omega^{45}$ generate a free Lie subalgebra. We begin by constructing a graphical spectral sequence with a Hopf algebra structure closely analogous to the structure on the Quillen spectral sequence produced in Theorem 3.18. The coproduct is induced by the Connes–Kreimer coproduct on the graphical Hopf algebra constructed in [CK98].

The graphical spectral sequence shall arise from the construction of a "K-theory of graphs," a bisimplicial space with the same structure and formal properties as $N_\bullet S_\bullet(\mathrm{Proj}_\mathbb{Z})$, leading to another spectral sequence of Hopf algebras. (Actually, for expository reasons, we shall consider three closely related variants. The first is most direct to define, while the last one is the one that shall be precisely related to the filtered space $BK(\mathbb{Z})$.) The E^1 page of this spectral sequence can be computed as homology of a suitable graph complex, and comes with a map (of spectral sequences of Hopf algebras) to the Quillen spectral sequence. The bisimplicial space we construct can be interpreted as a space of metric graphs.

Remark 5.1. These K-theory spaces of graphs, and their corresponding chain complexes, naturally have a cubical structure, and are reminiscent of earlier constructions of moduli spaces of metric graphs. We refer to Culler–Vogtmann's construction of Outer Space CV_n [CV86] and its marked-point variants $\mathrm{CV}_{n,s}$. These are used to study homology of the groups $\mathrm{Out}(F_n)$, $\mathrm{Aut}(F_n)$, and more generally a family of groups denoted $\Gamma_{n,s}$ generalizing them [Hat95, HV04, CHKV16]. The space CV_n is simplicial, but it has a deformation retraction to a subcomplex $\mathbb{S}\mathrm{CV}_n$ of its barycentric subdivision called the *spine*, which is admits the structure of a cubical space parametrizing marked metric graphs with chosen subforest, and has an associated graph complex [HV98, §3]. The spaces that we shall consider, denoted $|\overline{\mathcal{F}}_\bullet|$, $|\overline{\mathcal{F}}'_\bullet|$, and $|\overline{\mathcal{F}}''_\bullet|$ below, and their respective graph complexes, have similar structure.

Remark 5.2. There is a Waldhausen category $\mathcal{G}$, whose objects are all connected graphs G together with chosen basepoint $* \in V(G)$, morphisms all maps of graphs preserving base point, cofibrations the injective maps, and weak equivalences the ones whose corresponding map of topological spaces is a homotopy equivalence. There is also a symmetric monoidal structure given by wedge sum. Filtering by $b_1(G) \in \mathbb{N}$ induces a filtration of $BK(\mathcal{G})$ with the exact same formal properties as the filtration on $BK(\mathbb{Z})$, leading to a spectral sequence of Hopf algebras

$$E^1_{s,t} \Rightarrow BK(\mathcal{G}).$$

Moreover, the functor $H_1 \colon \mathcal{G} \to \mathrm{Proj}_{\mathbb{Z}}$ taking G to $H_1(G;\mathbb{Z})$ preserves all the structure and induces a map of spectral sequences of Hopf algebras.

At present we do not know how to make good use of this specific spectral sequence. Instead we proceed with a variant involving disconnected graphs without basepoint. It does not seem to literally fit into the axioms of a Waldhausen category, but we will explain an explicit construction which seems similar in spirit.

5.1. **K-theory of graphs.** We use the same definition of graphs as in [CGP21]: a graph is a finite set X, together with functions $i, r \colon X \to X$ that satisfy $i^2 = 1_X$ and $r^2 = r$, and such that

$$\{x \in X : r(x) = x\} = \{x \in X : i(x) = x\}.$$

Then we write $V = \{x \in X \mid i(x) = x\}$, $H = X \setminus V$, and $E = H/(x \sim i(x))$, interpreted as the set of vertices, the set of half-edges, and the set of edges of G. The most general notion of morphism from $G = (X, i, r)$ to $G' = (X', i', r')$ is just a map of sets $\phi : X \to X'$ such that $i' \circ \phi = i$ and $r' \circ \phi = r$, we do not require that it sends H to H'. Whenever we impose extra conditions on graphs (non-empty, connected, no zero-valent vertices, etc) of morphisms thereof, we will strive to say so explicitly. A morphism of graphs is injective (resp. surjective) if the underlying map on sets is injective (resp. surjective).

While officially there are no extra conditions on morphisms, in practice only a restricted class of morphisms will appear in the diagrams of graphs below. For example, surjective morphisms of graphs can in principle increase first Betti number by identifying vertices (for example, one may map the two endpoints of an edge to a single vertex, to create a self-edge). But such a morphism will never arise in the diagrams of graphs considered below.

Since we allow vertices of any valence including 0, the category of finite sets may be identified via $S \mapsto (S, \mathrm{id}, \mathrm{id})$ with a full subcategory of the category of graphs, namely those graphs G for which $E(G) = \emptyset$. Any graph G comes with universal maps to and from a set (more precisely, a map of graphs from G to a graph with no edges, initial among such, as well as a map of graphs from a graph with no edges to G, terminal among such) namely

$$V(G) \hookrightarrow G \twoheadrightarrow \pi_0(G),$$

where $V(G)$ is the set of vertices and $\pi_0(G)$ is the set of path components of G.

For expository reasons, we shall consider three possible variants of a K-theory space of graphs, corresponding to three simplicial categories $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ that we now define. For an object $[p] \in \Delta$, the three categories $\mathcal{F}_p$, $\mathcal{F}'_p$, and $\mathcal{F}''_p$ have the same object sets, namely the set of diagrams of the shape

$$
\begin{tikzcd}
G_{0,0} \arrow[r, tail] & G_{0,1} \arrow[r, tail] \arrow[d] & G_{0,2} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & G_{0,p} \arrow[d] \\
& G_{1,1} \arrow[r, tail] & G_{1,2} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & G_{1,p} \arrow[d] \\
& & {} & \vdots \arrow[d] & \vdots \arrow[d] \\
& & & G_{p-1,p-1} \arrow[r, tail] & G_{p-1,p} \arrow[d] \\
& & & & G_{p,p}
\end{tikzcd}
\tag{57}
$$

of graphs $G_{i,j}$ and morphisms between them, subject to the conditions

- $E(G_{i,i}) = \emptyset$ for $i = 0, \ldots, p$,
- the maps $V(G_{i,j}) \to V(G_{i,j+1})$ induced by the horizontal maps in the diagram are bijections, and $E(G_{i,j}) \to E(G_{i,j+1})$ are injections, $0 \leq i \leq j < p$,
- the maps $\pi_0(G_{i-1,j}) \to \pi_0(G_{i,j})$ induced by the vertical maps in the diagram are bijections,
- all squares in the diagrams are pushout diagrams.

The arrows denoted $\rightarrowtail$ in the diagram (57) are the ones that are required to be bijections on vertices and injections on edges. The conditions ensure that every vertical map is induced by contraction of edges only.

Remark 5.3.

(1) The diagonal maps $G_{i-1,i-1} \to G_{i,i}$ are surjections of finite sets. The diagram provides a factorization

$$G_{i-1,i-1} = V(G_{i-1,i-1}) \cong V(G_{i-1,i}) \twoheadrightarrow \pi_0(G_{i-1,i}) \cong \pi_0(G_{i,i}) \cong G_{i,i}$$

identifying $G_{i-1,i-1} \twoheadrightarrow G_{i,i}$ with the quotient by the equivalence relation generated by the edge set of $G_{i-1,i}$.

(2) Up to isomorphism, such a diagram is determined by its top row: because each rectangle in (57) of the form

$$
\begin{tikzcd}
G_{i-1,i} \arrow[r] \arrow[d] & G_{i-1,j} \arrow[d] \\
G_{i,i} \arrow[r] & G_{i,j}
\end{tikzcd}
$$

is a pushout diagram, each vertical map $G_{i-1,j} \to G_{i,j}$ must be induced by collapsing each edge in the image of $G_{i-1,i} \hookrightarrow G_{i-1,j}$ individually. The whole diagram can therefore be reconstructed up to isomorphism from the graph $G_{0,p}$ together with the flag of subsets of $E(G_{0,p})$ given by the images of $E(G_{0,j}) \hookrightarrow E(G_{0,p})$. As in the Waldhausen construction, it is, for set-theoretic reasons, convenient to include the data of chosen subquotients.

(3) By a similar argument, the diagram is also determined up to isomorphism by its rightmost column.

The three variants $\mathcal{F}_p$, $\mathcal{F}'_p$, and $\mathcal{F}''_p$ differ in the notion of morphisms between diagrams, which are defined as follows.

Definition 5.4.

(1) Let $\mathcal{F}_p$ be the groupoid whose objects are the diagrams (57), and whose morphisms are isomorphisms of such diagrams. Deleting the ith row and column define functors $d_i\colon \mathcal{F}_p \to \mathcal{F}_{p-1}$ for $0 \le i \le p$, and there are also functors $s_i\colon \mathcal{F}_p \to \mathcal{F}_{p+1}$ defined by inserting identities.

(2) Let $\mathcal{F}'_p$ be the category with the same object set as $\mathcal{F}_p$ but whose morphisms ϕ are maps of diagrams with components $\phi_{i,j}\colon G_{i,j} \to G'_{i,j}$ which are isomorphisms onto a subgraph $\phi_{i,j}(G_{i,j}) \subset G'_{i,j}$ containing all path components of positive genus (i.e. $G'_{i,j}$ is the disjoint union of $\phi_{i,j}(G_{i,j})$ and a forest) and such that for all $i < j$ the diagram

$$\begin{tikzcd} G_{0,i} \arrow[r] \arrow[d, "\phi_{0,i}"'] & G_{0,j} \arrow[d, "\phi_{0,j}"] \\ G'_{0,i} \arrow[r] & G'_{0,j} \end{tikzcd} \tag{58}$$

is cartesian.

(3) Let $\mathcal{F}''_p$ be the category with the same object set as $\mathcal{F}_p$ but whose morphisms ϕ are maps of diagrams with components $\phi_{i,j} : G_{i,j} \to G'_{i,j}$ such that $\phi_{0,p}^{-1}(V(G'_{0,p})) = V(G_{0,p})$, i.e. $\phi_{0,p}$ does not collapse any edge to a vertex, such that the induced map $E(G_{0,p}) \to E(G'_{0,p})$ is injective, such that the induced map $H_1(G_{0,p}) \to H_1(G'_{0,p})$ is an isomorphism, and such that the diagrams (58) are cartesian.

The condition in (3) that $H_1(\phi_{0,p})$ be an isomorphism amounts to the following: firstly, if two distinct vertices $v, v' \in V(G_{0,p})$ are mapped to the same vertex $\phi_{0,p}(v) = \phi_{0,p}(v') \in V(G'_{0,p})$ then v and v' must be in different path components of $G_{0,p}$; secondly, if an edge $e \in E(G'_{0,p})$ is not in the image of the injection $E(\phi_{0,p})$, then e is a bridge in $G'_{0,p}$.

The significance for requiring cartesianness of the diagram (58) is that morphisms in these categories are determined by the component $\phi_{0,p} : G_{0,p} \to G'_{0,p}$, the map of "upper right corners" in the triangular diagrams. The induced injection of sets

$$E(\phi_{0,p}) : E(G_{0,p}) \hookrightarrow E(G'_{0,p})$$

must satisfy that the flag of sets (59) is the pullback of the corresponding flag of subsets of $E(G'_{0,p})$. This implies for $0 \le j \le p$ that the map of graphs $\phi_{0,j} : G_{0,j} \to G'_{0,j}$ also does not collapse an edge to a vertex and that $E(\phi_{0,j}) : E(G_{0,j}) \to E(G'_{0,j})$ is injective. Finally, we may also conclude that $H_1(\phi_{0,j}) : H_1(G_{0,j}) \to H_1(G'_{0,j})$ is an isomorphism: this is because any element $e \in E(G'_{0,p})$ not in the image of $E(\phi_{0,p}) : E(G_{0,p}) \hookrightarrow E(G'_{0,p})$ must be a bridge (otherwise the map could not induce an isomorphism on H_1), and then e must also be a bridge in any subgraph of $G'_{0,p}$ containing e. It follows that the conditions imposed on $\phi_{0,p} : G_{0,p} \to G'_{0,p}$

in (3) are also satisfied by $\phi_{0,i} : G_{0,i} \to G'_{0,i}$. By a similar argument, they are also satisfied by $\phi_{i,j} : G_{i,j} \to G'_{i,j}$ for all $i \leq j$, using that $G_{i,j}$ is the quotient of $G_{0,j}$ by collapsing the edges in $E(G_{0,i})$ and similarly for $G'_{i,j}$.

This discussion shows that the condition in (3) is preserved by the face and degeneracy maps in $[p] \mapsto \mathcal{F}''_p$, justifying that this indeed defines a simplicial object in the category of small categories. Similarly for $[p] \mapsto \mathcal{F}'_p$ and $[p] \mapsto \mathcal{F}_p$, the latter of which is even a simplicial object in small groupoids. There are functors

$$\mathcal{F}_p \hookrightarrow \mathcal{F}'_p \hookrightarrow \mathcal{F}''_p \xrightarrow{H_1} S_p(\mathrm{Proj}_{\mathbb{Z}}),$$

where the first two functors are inclusions of subcategories (relaxing the conditions on morphisms of diagrams) and the third functor is induced by sending $G_{i,j} \mapsto H_1(G_{i,j}; \mathbb{Z})$. The first two functors are bijections on object sets and on isomorphism sets, but $\mathcal{F}'_p$ and $\mathcal{F}''_p$ have more non-invertible morphisms. Up to unique isomorphism, an object of these categories is determined by the "upper right" entry $G_{0,p}$ in the diagram (57) together with a flag of sets

$$\emptyset = E(G_{0,0}) \hookrightarrow E(G_{0,1}) \hookrightarrow \cdots \hookrightarrow E(G_{0,p}). \tag{59}$$

These functors induce maps of bisimplicial sets and in turn of topological spaces

$$N_\bullet \mathcal{F}_\bullet \hookrightarrow N_\bullet \mathcal{F}'_\bullet \hookrightarrow N_\bullet \mathcal{F}''_\bullet \xrightarrow{H_1} N_\bullet S_\bullet(\mathrm{Proj}_{\mathbb{Z}}), \tag{60}$$

$$|N_\bullet \mathcal{F}_\bullet| \hookrightarrow |N_\bullet \mathcal{F}'_\bullet| \hookrightarrow |N_\bullet \mathcal{F}''_\bullet| \longrightarrow |N_\bullet S_\bullet(\mathrm{Proj}_{\mathbb{Z}})| = BK(\mathbb{Z}). \tag{61}$$

Remark 5.5. For set-theoretic reasons, the above definitions should be augmented with a choice of small category $\mathcal{G}$ equivalent to all graphs and all morphisms (for instance by insisting $G = (X, i, r)$ where $X \subset \Omega$ is a subset of some fixed infinite set, or by choosing a graph of each isomorphism type). Then we get small categories $\mathcal{F}_p$, $\mathcal{F}'_p$, and $\mathcal{F}''_p$ whose object set is the set of diagrams in $\mathcal{G}$ of the form (57), subject to the stated requirements.

5.2. **Filtrations and graphical spectral sequences.**

Definition 5.6.

- *For $g \in \mathbb{N}$ let*
$$F_g\mathcal{F}_p \subset \mathcal{F}_p, \quad F_g\mathcal{F}'_p \subset \mathcal{F}'_p, \quad F_g\mathcal{F}''_p \subset \mathcal{F}''_p$$
be the full subcategories containing those objects (57) with $b_1(G_{0,p}) \leq g$, or equivalently that $b_1(G_{i,j}) \leq g$ for all i, j.
- *Let $F_g|N_\bullet\mathcal{F}_\bullet| \subset |N_\bullet\mathcal{F}_\bullet|$ be the image of the map induced by $N_\bullet(F_g\mathcal{F}_\bullet) \to N_\bullet\mathcal{F}_\bullet$, the nerve of the inclusion of the subcategory, and similarly for $F_g|N_\bullet\mathcal{F}'_\bullet| \subset |N_\bullet\mathcal{F}'_\bullet|$ and $F_g|N_\bullet\mathcal{F}''_\bullet| \subset |N_\bullet\mathcal{F}''_\bullet|$.*

We have now defined filtrations on all four spaces in (61), and it is clear that the maps preserve filtrations. Therefore there are induced maps of spectral sequences

$$^{G}E^r_{*,*} \to {}^{G'}E^r_{*,*} \to {}^{G''}E^r_{*,*} \to {}^{Q}E^r_{*,*}, \tag{62}$$

converging to the rational homology of the spaces (61). We will call the first three spectral sequences the "graph spectral sequences" since their definition is based on graphs, and speak of the un-primed/primed/double-primed variants when it is relevant to distinguish them.

Proposition 5.7. *The construction in Section 3 apply to the bisimplicial set $X_{p,q} = N_q\mathcal{F}_p$ filtered as $F_gX_{p,q} = N_q(F_g\mathcal{F}_p)$, leading to a space-level coproduct*

$$|X| \xrightarrow{\approx} |\mathrm{es}(X)| \to |X| \times |X|.$$

which is a filtered map.

Choosing a disjoint union operation $(G, G') \mapsto m(G, G') \cong G \sqcup G'$ on the chosen small category equivalent to all finite graphs and promoting it to a symmetric monoidal functor, leads to a map of bisimplicial sets $X \times X \to X$ inducing a product

$$m : |X| \times |X| \xrightarrow{\approx} |X \times X| \to |X|.$$

The product and coproduct are both filtered maps, the product is associative and commutative up to a filtration-preserving homotopy, and the coproduct is associative up to a filtration-preserving homotopy. Forgetting filtrations, the coproduct is homotopic to the diagonal map of $|X|$.

The same is true for the two filtered bisimplicial spaces $X' = N_\bullet\mathcal{F}'_\bullet$ and $X'' = N_\bullet\mathcal{F}''_\bullet$, and the induced maps $|X| \to |X'| \to |X''|$ are compatible with coproduct, product, and filtration.

Lemma 5.8. *Let*

$${}^{G}E^1_{*,*} \Rightarrow H_*(|N_\bullet\mathcal{F}_\bullet|)$$

be the homological spectral sequence associated to the filtered space defined above. This admits the structure of a spectral sequence of bialgebras, with graded commutative product on all pages and graded co-commutative co-product on the E^∞ page.

Similarly for the primed and double-primed variants. Furthermore, the maps of spectral sequences induced by (61) are bialgebra maps on each page.

As we will see later, the un-primed graph spectral sequence will have $E^1_{0,0} \cong \mathbb{Q}[x]/(x^2 - x)$, with unit 1 the class of the empty graph and $x \in E^1_{0,0}$ the class of a graph consisting of a single zero-valent vertex. In this bidegree the coproduct is given by $\Delta(x) = x \otimes x$ and the augmentation by $x \mapsto 1$. This bigraded bialgebra is not connected and the existence of an antipode is not automatic, hence "bialgebra" instead of "Hopf algebra" in the above statement. (And indeed it does not admit an antipode: recall that Spec of a commutative bialgebra is a monoid scheme, and is a group scheme if and only if the bialgebra is part of a Hopf algebra structure. See, e.g., [Mil12, Theorem 5.1]. The bialgebra $E^1_{0,0}$ represents the functor sending a commutative ring R to its monoid of idempotent elements—this defines a monoid scheme $\mathrm{Spec}(\mathbb{Q}[x]/(x^2 - x))$ which is not a group scheme). We will also see later that the element $x \in {}^{G}E^1_{0,0}$ maps to a unit in the primed version of the graph spectral sequence, and consequently the other basis element $1 - x \in {}^{G}E^1_{0,0}$ must map to zero. The (bi)graded bialgebras in the pages of the primed and double-primed spectral sequences are in fact connected, and hence admit unique antipodes making them Hopf algebras. For clarity it seems cleanest to present the arguments for $\mathcal{F}_\bullet$ first, dealing with the (easy) modifications in the primed and double-primed variants later.

Proof sketch. The coproduct is constructed in the same way as for $BK(\mathbb{Z})$ and uses no structure besides the filtration and the bisimplicial structure. The coassociativity of the coproduct, up to a homotopy respecting the filtration, follows from Proposition 3.11 and the observation that given

$$G_0 \rightarrowtail \cdots \rightarrowtail G_{2p+1},$$

where the arrows $\rightarrowtail$ are bijections on vertices and injections on edges, the inequality

$$b_1(G_{2p+1}) \geq b_1(G_p) + b_1(G_{2p+1}/G_{p+1})$$

holds. From this observation, it follows that

$$\mathrm{es}(F_s(N_\bullet F_\bullet)) \Rightarrow F_s(N_\bullet F_\bullet \times N_\bullet F_\bullet) \Rightarrow F_s(N_\bullet F_\bullet) \times F_s(N_\bullet F_\bullet),$$

so that the hypotheses of Proposition 3.11 are satisfied.

The product on $S_p(\mathrm{Proj}_\mathbb{Z})$ came from the symmetric monoidal structure induced by direct sum, but we can define a symmetric monoidal structure on $\mathcal{F}_p$ by disjoint union of graphs, and clearly $H_1 \colon \mathcal{F}_p \to S_p(\mathrm{Proj}_\mathbb{Z})$ promotes to a symmetric monoidal functor.

The primed and double-primed variants are handled by the same argument. Moreover, it is clear that the maps (61) preserve all structure in sight. □

In the remainder of this section, we explain why a filtered space rationally equivalent to $|N_\bullet \mathcal{F}_\bullet|$ may be interpreted as a moduli space of possibly-disconnected metric graphs, and explain how the E^1 page may be identified with homology of a certain graph complex.

Definition 5.9. *Let $\overline{\mathcal{F}}_p$ denote the set of isomorphism classes in the groupoid $\mathcal{F}_p$, in other words the coequalizer of $d_0, d_1 : N_1\mathcal{F}_p \to N_0\mathcal{F}_p$ in the category of sets, or the set $\pi_0(|N_\bullet\mathcal{F}_p|)$ of path components of the geometric realization of the nerve of $\mathcal{F}_p$.*

Similarly, let $\overline{\mathcal{F}}'_p$ be the coequalizer of $d_0, d_1 : N_1\mathcal{F}'_p \to N_0\mathcal{F}'_p$ and $\overline{\mathcal{F}}''_p$ the coequalizer of $d_0, d_1 : N_1\mathcal{F}''_p \to N_0\mathcal{F}''_p$.

The sets $\overline{\mathcal{F}}_p$ assemble into a simplicial set $\overline{\mathcal{F}}_\bullet$. Regarding $\overline{\mathcal{F}}_p$ as a space with discrete topology, the canonical maps

$$|N_\bullet \mathcal{F}_p| \to \overline{\mathcal{F}}_p \tag{63}$$

for all p assemble to a map of simplicial spaces, with geometric realization

$$|N_\bullet \mathcal{F}_\bullet| \to |\overline{\mathcal{F}}_\bullet|. \tag{64}$$

The space $|\overline{\mathcal{F}}_\bullet|$ does not itself map to $BK(\mathbb{Z})$, but is a simpler object in that it arises from a simplicial set as opposed to a bisimplicial set. Furthermore, it is a good model up to rational equivalence:

Lemma 5.10. *The maps (63) and (64) induce isomorphisms in rational homology. Filtering $\overline{\mathcal{F}}_p$ by the images of the filtration in $N_0\mathcal{F}_p$, the map (64) also induces an isomorphism in rational homology of associated gradeds.*

Proof. The homotopy type of the domain of (63) is a disjoint union of $K(\pi, 1)$'s for the automorphism groups of diagrams of the form (57), one for each isomorphism class of such diagrams. Since any finite group has the rational homology of a point, the map (63) is a rational equivalence for each p. Filtering the realization in the p-direction by skeleta shows that (64) is too. □

Similarly for the primed and double-primed versions, using the following.

Lemma 5.11. *The maps*

$$|N_\bullet \mathcal{F}'_\bullet| \to |\overline{\mathcal{F}}'_\bullet|$$
$$|N_\bullet \mathcal{F}''_\bullet| \to |\overline{\mathcal{F}}''_\bullet|,$$

induced by sending a q-simplex $x \in N_q(\mathcal{F}'_p)$ to the path component $[x] \in \pi_0(N_\bullet \mathcal{F}'_p) = \overline{\mathcal{F}}'_p$ containing it, and similarly for $\mathcal{F}''_\bullet$, induce isomorphisms in rational homology.

Proof. As in the proof of Lemma 5.10, it suffices to see that $|N_\bullet \mathcal{F}'_p|$ has the homotopy type of a disjoint union of $K(\pi, 1)$-spaces for finite groups π, and similarly for $|N_\bullet \mathcal{F}''_p|$. The problem is that this is no longer an immediate consequence of finiteness of Hom-spaces in these categories. Instead we argue as follows.

Let $C'_p \subset \mathcal{F}'_p$ be the full subcategory on those diagrams in which all path components of $G_{0,p}$ have positive genus. Inspecting the definition of morphism in $\mathcal{F}'_p$, one convinces oneself that all morphisms in C'_p are isomorphisms, in other words C'_p is a groupoid. Moreover, the inclusion functor $i\colon C'_p \hookrightarrow \mathcal{F}'_p$ admits a right adjoint $R\colon \mathcal{F}'_p \to C'_p$, given by removing all genus-zero components of $G_{0,p}$ and their images and inverse images in the other $G_{i,j}$. It is well known that the adjunction implies that the maps

$$|N_\bullet i|\colon |N_\bullet C'_p| \to |N_\bullet \mathcal{F}'_p|$$
$$|N_\bullet R|\colon |N_\bullet \mathcal{F}'_p| \to |N_\bullet C'_p|$$

are inverse homotopy equivalences. Since C'_p is a groupoid with finite automorphism groups, the geometric realization of the nerve is a disjoint union of $K(\pi, 1)$-spaces for finite groups π. As in the case of $\mathcal{F}_p$, we deduce that the geometric realization of its nerve is equivalent to the discrete space $\pi_0(N_\bullet \mathcal{F}'_p)$.

An entirely similar argument applies to $\mathcal{F}''_p$, using the full subcategory $C''_p \subset \mathcal{F}''_p$ on those diagrams in which all components of $G_{0,p}$ have positive genus, all edges in $G_{0,p}$ are contained in a cycle, and no vertex is a cut vertex. In this case, the right adjoint to the inclusion functor deletes all edges and vertices not contained in a cycle, and separates cut vertices.

(Note that the larger categories $\mathcal{F}'_p$ and $\mathcal{F}''_p$ are still convenient to keep around, since the face maps do not restrict to functors $C'_p \to C'_{p-1}$ and $C''_p \to C''_{p-1}$.) □

5.3. **A rational cell decomposition of $|\overline{\mathcal{F}}_\bullet|$.** The geometric realization $|\overline{\mathcal{F}}_\bullet|$ can be described rather explicitly, as an iterated "rational cell attachment." These cells most naturally have cubical shape, attaching the cube $(\Delta^1)^n$ to a space X along an attaching map $e : \partial(\Delta^1)^n \to X$, and more generally as a pushout of the form

$$(\Delta^1)^n/H \leftarrow \partial(\Delta^1)^n/H \xrightarrow{e} X$$

for a subgroup $H < S_n$ acting on the cube and its boundary by permuting coordinates. When $H \leq A_n$ then $\partial(\Delta^1)^n/H$ is a rational homology S^{n-1} and the attachment has the same effect in rational homology as attaching an ordinary n-cell in the usual sense. When H contains an odd permutation, then the attachment does not change the rational homology. As mentioned in Remark 5.1, this description of $|\mathcal{F}_\bullet|$ as a cubical space of graphs, and the graph complex we shall associate to it in Section 5.4, is a variant of the cubical spaces of graphs introduced by Hatcher–Vogtmann [HV98, §3] in their work computing rational homology groups of $\mathrm{Aut}(F_n)$.

It is also similar in spirit, although not isomorphic, to the notion of "symmetric Δ-complex" in [CGP21]: in that paper we used iterated attachment of the pair $\partial\Delta^{n-1}/H \subset \Delta^{n-1}/H$, a rational homology $(n-1)$-disk, instead of the pair $\partial(\Delta^1)^n/H \subset (\Delta^1)^n/H$, a rational homology n-disk.

The key observation is that any graph G gives rise to a somewhat canonical map of simplicial sets

$$(\Delta^1_\bullet)^{E(G)} \to N_0(\mathcal{F}_\bullet) \to \overline{\mathcal{F}}_\bullet,$$

where $\Delta^1_\bullet$ is the usual representable simplicial set $[p] \mapsto \Delta([p],[1])$ with geometric realization Δ^1. We now explain this canonical map precisely. Given a p-simplex $(f_e)_{e\in E(G)}$ of $(\Delta^1_\bullet)^{E(G)}$, where $f_e \in \Delta([p],[1])$, we associate the sequence of subsets of $E(G)$

$$\emptyset \subset E_0 \subset \cdots \subset E_p \subset E(G) \tag{65}$$

in which

$$E_j = \{e \in E(G) : f_e(j) = 1\}. \tag{66}$$

We may now form a diagram of graphs of the form (57), by first writing $G_{-1,j}$ for the graph with $V(G_{-1,j}) = V(G)$ and $E(G_{-1,j}) = E_j$, and then for $0 \le i \le j \le p$ defining $G_{i,j}$ as the quotient of $G_{-1,j}$ obtained by collapsing those edges $e \in E(G_{-1,j}) = E_j$ which are elements of the subset $E_i \subset E_j$. This recipe gives a diagram of the form (57) satisfying all requirements, except that the chosen quotient graphs $G_{i,j}$ may not literally be in the object set of $\mathcal{G}$. (Recall that we chose a small category in order for $N_\bullet\mathcal{F}_\bullet$ to be a bisimplicial *set*.) Each such diagram is certainly *isomorphic* to a diagram in $\mathcal{G}$, and choosing an isomorphic diagram in $\mathcal{G}$ for each p and each $f \in \Delta^1_p$ produces a map of simplicial sets $(\Delta^1_\bullet)^{E(G)} \to N_0\mathcal{F}_\bullet$. The map $(\Delta^1_\bullet)^{E(G)} \to N_0(\mathcal{F}_\bullet)$ constructed in this way depends on choices, but the composition $(\Delta^1_\bullet)^{E(G)} \to N_0(\mathcal{F}_\bullet) \to \overline{\mathcal{F}}_\bullet$ does not, and factors over the quotient $(\Delta^1_\bullet)^{E(G)}/\mathrm{Aut}(G)$. We have produced commutative diagrams

$$\begin{tikzcd}
(\Delta^1_\bullet)^{E(G)} \arrow[r] \arrow[d] & N_0(\mathcal{F}_\bullet) \arrow[d] \\
(\Delta^1_\bullet)^{E(G)}/\mathrm{Aut}(G) \arrow[r] & \overline{\mathcal{F}}_\bullet.
\end{tikzcd}$$

Recall that the geometric realization functor preserves small colimits and finite products, and that $|\Delta^1_\bullet| = \Delta^1$, so the induced maps of geometric realizations may be written as

$$(\Delta^1)^{E(G)}/\mathrm{Aut}(G) \xleftarrow{\approx} |(\Delta^1_\bullet)^{E(G)}/\mathrm{Aut}(G)| \longrightarrow |\overline{\mathcal{F}}_\bullet|. \tag{67}$$

The following lemma summarizes the sense in which these canonical maps $(\Delta^1)^{E(G)}/\mathrm{Aut}(G) \to |\overline{\mathcal{F}}_\bullet|$ behave (rationally) like cells of a CW structure. It also defines an analogue of the filtration by skeleta (not to be confused with the more interesting filtration by first Betti number, which leads to the graph spectral sequence).

Lemma 5.12. *Let $\overline{\mathcal{F}}^{(n)}_\bullet \subset \overline{\mathcal{F}}_\bullet$ denote the simplicial subset whose p-simplices are cut out by the condition that the cardinality of $E(G_{0,p})$ is at most n. Then the maps (67) assemble to pushout* *diagrams of topological spaces*

$$\begin{tikzcd} \coprod_G \partial(\Delta^1)^{E(G)}/\mathrm{Aut}(G) \arrow[r] \arrow[d] & \lvert\overline{\mathcal{F}}_\bullet^{(n-1)}\rvert \arrow[d] \\ \coprod_G (\Delta^1)^{E(G)}/\mathrm{Aut}(G) \arrow[r] & \lvert\overline{\mathcal{F}}_\bullet^{(n)}\rvert, \end{tikzcd}$$

where the coproduct is indexed by graphs G with $|E(G)| = n$, one in each isomorphism class of such graphs.

The entry $\partial(\Delta^1)^{E(G)}/\mathrm{Aut}(G)$ in the diagram should be parsed as $\big(\partial\big((\Delta^1)^{E(G)}\big)\big)/\mathrm{Aut}(G)$: the orbit space of the action of $\mathrm{Aut}(G)$ on the boundary of the cube $(\Delta^1)^{E(G)}$.

Using this presentation as an iterated pushout, we deduce descriptions of the associated graded with respect to this filtration of $|\mathcal{F}_\bullet|$ by number of edges, as well as a description of the underlying point set.

Corollary 5.13.

(1) *The maps (67) induce homeomorphisms of pointed spaces*

$$\bigvee_{|E(G)|=n} (S^1)^{\wedge E(G)}/\mathrm{Aut}(G) \longrightarrow \frac{|\overline{\mathcal{F}}_\bullet^{(n)}|}{|\overline{\mathcal{F}}_\bullet^{(n-1)}|}$$

where the coproduct of pointed spaces is indexed by graphs with precisely n edges, one in each isomorphism class, and $S^n \approx (S^1)^{\wedge E(G)}$ is the $E(G)$-fold smash product of the circle with itself, on which $\mathrm{Aut}(G)$ *acts by permuting factors.*

(2) *The maps (67) induce a continuous bijection*

$$\coprod_G (\Delta^1 \setminus \partial\Delta^1)^{E(G)}/\mathrm{Aut}(G) \longrightarrow |\overline{\mathcal{F}}_\bullet|,$$

where the disjoint union is over all graphs, one in each isomorphism class.

Remark 5.14. The second part of the corollary shows that the set underlying $|\overline{\mathcal{F}}_\bullet|$ may be interpreted as the set of isomorphism classes of pairs (G,ℓ) where $\ell : E(G) \to \Delta^1 \setminus \partial\Delta^1$ is any function, and $[G,\ell] = [G',\ell']$ if there exists an isomorphism $\phi : G \to G'$ such that $\ell'(\phi(e)) = \ell(e)$ for all $e \in E(G)$. For psychological reasons it may be preferable to rescale using a homeomorphism $\Delta^1 \setminus \partial\Delta^1 \to (0,\infty)$ such as $(t_0,t_1) \mapsto -\log(t_1)$. This gives an interpretation of $|\overline{\mathcal{F}}_\bullet|$ as a moduli space of metric graphs where edges have assigned positive real lengths. In this picture, the topology may described informally as “edges are collapsed as their length go to zero, and edges are deleted as their length go to infinity”.

Proof of Lemma 5.12. Inspecting the definition of the canonical map (67), we see that it factors through the subspace $|\overline{\mathcal{F}}_\bullet^{(n)}|$ when $|E(G)| \leq n$ and that for each $e \in E(G)$, its composition with

$$(\partial\Delta^1)^{\{e\}} \times (\Delta^1)^{E(G)\setminus\{e\}} \hookrightarrow (\Delta^1)^{E(G)} \twoheadrightarrow (\Delta^1)^{E(G)}/\mathrm{Aut}(G)$$

will factor through $|\overline{\mathcal{F}}_\bullet^{(n-1)}|$ because the only graphs that will appear are subquotients of $G \setminus e$ and G/e, so they will have strictly fewer edges than G has. This observation explains why (67) induce horizontal maps in the square diagram in the lemma, and it is then clear that the diagram commutes.

To prove that the diagram is pushout, it suffices to consider the corresponding diagram of simplicial sets, again because geometric realization preserves small colimits. Pushouts in simplicial sets are computed degree-wise, so it suffices to prove that the diagram of p-simplices

$$\begin{array}{ccc} \coprod_G (\partial(\Delta^1_\bullet)^{E(G)})_p/\mathrm{Aut}(G) & \longrightarrow & \overline{\mathcal{F}}^{(n-1)}_p \\ \downarrow & & \downarrow \\ \coprod_G (\Delta^1_p)^{E(G)}/\mathrm{Aut}(G) & \longrightarrow & \overline{\mathcal{F}}^{(n)}_p \end{array}$$

is pushout in sets. It is easy to see that both vertical maps are injective, so we must show that the horizontal map restricts to a bijection between the complements of the images of the vertical maps.

Before passing to $\mathrm{Aut}(G)$-orbits, the subset $(\partial(\Delta^1_\bullet)^{E(G)})_p \subset (\Delta^1_p)^{E(G)} = \Delta([p],[1])^{E(G)}$ correspond to those poset maps $[p] \to [1]^{E(G)}$ where at least one coordinate is constant. The complement is therefore precisely the set

$$(\Delta^1_p)^{E(G)} \setminus (\partial(\Delta^1_\bullet)^{E(G)})_p = \Delta_{\mathrm{surj}}([p],[1])^{E(G)}$$

consisting of $E(G)$-tuples of morphisms $[p] \to [1]$ in the simplex category Δ whose underlying map of sets is surjective.

The proof is concluded by firstly observing that the set map $(\Delta^1_p)^{E(G)} \to \overline{\mathcal{F}}_p$ defined above will send $\Delta_{\mathrm{surj}}([p],[1])^{E(G)}$ into $\overline{\mathcal{F}}^{(n)}_p \setminus \overline{\mathcal{F}}^{(n-1)}_p$ if $|E(G)| = n$: indeed, in this case the flag of subsets (65) of $E(G)$ will have $E_0 = \emptyset$ and $E_p = E(G)$ and hence $G_{0,p} = G$. Secondly, the resulting map of sets

$$\coprod_{|E(G)|=n} \Delta_{\mathrm{surj}}([p],[1])^{E(G)}/\mathrm{Aut}(G) \to \overline{\mathcal{F}}^{(n)}_p \setminus \overline{\mathcal{F}}^{(n-1)}_p$$

is surjective because any diagram of the form (57) with $|E(G_{0,p})| = n$ arises from $G = G_{0,p}$ and the flag of subsets $\emptyset = E_0 \subset \cdots \subset E_p = E(G)$ such that $E(G_{0,j}) = E_j$, corresponding to some unique $f = (f_e)_{e \in E(G)} \in \Delta_{\mathrm{surj}}([p],[1])^{E(G)}$. Conversely the map is injective because this flag of subsets of $E(G)$ is uniquely determined up to automorphisms of G. $\square$

By entirely similar arguments, we deduce an interpretation of the primed version $|\overline{\mathcal{F}}'_\bullet|$ as a moduli space of isometry classes of metric graphs where one furthermore identifies graphs that differ by disjoint union with genus-zero graphs. For the double-primed version $|\overline{\mathcal{F}}''_\bullet|$ we deduce an analogue of the pushout presentation in Lemma 5.12, by an analogous argument.

Lemma 5.15. *Let $\overline{\mathcal{F}}_p''^{(n)} \subset \overline{\mathcal{F}}_p''$ be the image of $\overline{\mathcal{F}}_p^{(n)}$ under the quotient map $\overline{\mathcal{F}}_p \to \overline{\mathcal{F}}_p''$. Then the maps (67) assemble to pushout diagrams*

$$\begin{tikzcd} \coprod_G \partial(\Delta^1)^{E(G)}/\mathrm{Aut}(G) \arrow[r] \arrow[d] & |\overline{\mathcal{F}}_\bullet''^{(n-1)}| \arrow[d] \\ \coprod_G (\Delta^1)^{E(G)}/\mathrm{Aut}(G) \arrow[r] & |\overline{\mathcal{F}}_\bullet''^{(n)}|, \end{tikzcd}$$

where the disjoint union is indexed by graphs G with $|E(G)| = n$, which furthermore have no bridges, acyclic path components, or cut vertices, one in each isomorphism class of such graphs. □

Corollary 5.16. *(1) The maps (67) induce homeomorphisms of pointed spaces*

$$\bigvee_{|E(G)|=n} (S^1)^{\wedge E(G)}/\mathrm{Aut}(G) \longrightarrow \frac{|\overline{\mathcal{F}}_\bullet''^{(n)}|}{|\overline{\mathcal{F}}_\bullet''^{(n-1)}|}$$

where the coproduct of pointed spaces is indexed by graphs with precisely n edges, which furthermore have no bridges, acyclic path components, or cut vertices, one in each isomorphism class.

(2) The maps (67) induce a continuous bijection

$$\coprod_G (\Delta^1 \setminus \partial\Delta^1)^{E(G)}/\mathrm{Aut}(G) \longrightarrow |\overline{\mathcal{F}}_\bullet''|,$$

where the disjoint union is over all graphs, which furthermore have no bridges, acyclic path components, or cut vertices, one in each isomorphism class.

□

5.4. **A graph complex modeling $C_*(\overline{\mathcal{F}}_\bullet)$.** If G is a graph with $|E(G)| = n$ then any choice of bijection $\omega : \{0, \dots, n-1\} \to E(G)$ induces a homeomorphism $S^n = (S^1)^{\wedge n} \approx (S^1)^{\wedge E(G)}$ and induces a group homomorphism $\mathrm{Aut}(G) \to S_n$. Let us write $H = H_{G,\omega} < S_n$ for the image of this group homomorphism. Then ω also induces a homeomorphism

$$(S^1)^{\wedge E(G)}/\mathrm{Aut}(G) \xrightarrow{\cong} S^n/H$$

and the quotient map $S^n \to S^n/H$ is a rational homology isomorphism if $H < A_n$, whereas S^n/H has the rational homology of a point when H contains an odd permutation. To the eyes of rational homology, the filtration of $|\overline{\mathcal{F}}_\bullet|$ by number of edges therefore behaves like a CW structure whose n-cells are in bijection with the set of isomorphism classes of graph G with $|E(G)| = n$ and such that G does not admit any automorphisms inducing an odd permutation of $E(G)$. This leads to a "cellular chain complex" quasi-isomorphic to $C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|; \mathbb{Q})$. This cellular chain complex has a combinatorial description which we spell out, as well as the filtration corresponding to first Betti number. The latter gives a "graph complex" point of view on the spectral sequence of Hopf algebras mapping to the Quillen spectral sequence.

Let $C = (C_*, \partial)$ be the rational chain complex generated by symbols $[G, \omega]$, where G is any graph and ω is a total order on $E(G)$, subject to the usual relation $[G, \omega] = \mathrm{sgn}(\sigma)[G', \omega']$ for each isomorphism $G \to G'$ with respect to which ω and ω' are related by permutation σ. We do not impose any conditions whatsoever on $G = (X, i, r)$: in particular, we allow disconnected graphs, bridges, loops, parallel edges, and vertices of any valence including zero.

The degree of the generator $[G, \omega]$ is declared to be $|E(G)|$, and we write $C_p \subset C$ for the span of those $[G, \omega]$ for which $|E(G)| = p$. We will return to the boundary map shortly, but first we define the comparison map to $C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|; \mathbb{Q})$. To this end, recall that the identity map of Δ^1, regarded as a chain $\iota_1 \in C_1(\Delta^1; \mathbb{Z})$, represents the fundamental class in $H_1(\Delta^1, \partial\Delta^1; \mathbb{Z})$. Then we use the cross product ([Hat02, p.277–278]) natural transformation $C_p(X) \otimes C_q(Y) \to C_{p+q}(X \times Y)$ to define a system of fundamental chains

$$\iota_n \in C_n((\Delta^1)^n; \mathbb{Z}), \tag{68}$$

compatible in the sense that $\iota_n \times \iota_m = \iota_{n+m}$. There is also an explicit formula for ι_n as a signed sum of $n!$ many maps $\Delta^n \to (\Delta^1)^n$, corresponding to all the non-degenerate n-simplices of $(\Delta^1_\bullet)^n$ with appropriate signs, but we will not need this explicit formula.

Definition 5.17. *For a graph G with $|E(G)| = n$ and a total ordering $\omega : \{0, \ldots, n-1\} \to E(G)$, write $f^{G,\omega} : (\Delta^1)^n \to |\overline{\mathcal{F}}_\bullet|$ for the composition*

$$(\Delta^1)^n \approx (\Delta^1)^{E(G)} \to |\overline{\mathcal{F}}_\bullet|$$

of the homeomorphism induced by the bijection ω, followed by the canonical map defined as in (67). Then define the linear map

Let

$$\begin{aligned} C_p &\to C_p^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|; \mathbb{Q}). \\ [G, \omega] &\mapsto f_*^{G,\omega}(\iota_n) \end{aligned} \tag{69}$$

With respect to the filtration by number of edges, we deduce the following from Corollary 5.13.

Lemma 5.18. *The relative homology*

$$H_*\big(\big|\overline{\mathcal{F}}_\bullet^{(p)}\big|, \big|\overline{\mathcal{F}}_*^{(p-1)}\big|; \mathbb{Q}\big)$$

vanishes for $* \neq p$. If $|E(G)| = p$, then the homomorphism (69) sends $[G, \omega]$ into the subspace $C_p(|\overline{\mathcal{F}}_*^{(p)}|; \mathbb{Q})$. Its image in relative chains

$$f_*^{G,\omega}(\iota_n) \in C_p\big(\big|\overline{\mathcal{F}}_*^{(p)}\big|, \big|\overline{\mathcal{F}}_*^{(p-1)}\big|; \mathbb{Q}\big)$$

is a cycle, and the resulting map of rational vector spaces

$$\begin{aligned} C_p &\to H_p\big(\big|\overline{\mathcal{F}}_*^{(p)}\big|, \big|\overline{\mathcal{F}}_*^{(p-1)}\big|; \mathbb{Q}\big) \\ [G, \omega] &\to \big[f_*^{G,\omega}(\iota_n)\big] \end{aligned}$$

is an isomorphism.

Proof sketch. This is an easy consequence of the description of $\big|\overline{\mathcal{F}}^{(p)}\big| / \big|\overline{\mathcal{F}}^{(p-1)}\big|$ in Corollary 5.13, together with the fact that $[\iota_n] \in H_n((\Delta^1)^n/H, \partial(\Delta^1)^n/H; \mathbb{Q})$ is a generator, for any subgroup $H \leq A_n$. □

We have not yet defined the boundary operator $\partial : C_p \to C_{p-1}$, but let us do that now. It has two terms $\partial = \partial_d - \partial_c$, one which deletes an edge in all possible ways, and one which contracts an edge in all possible ways. Precisely, for G a graph with $p+1$ edges and $\omega = e_0 < \cdots < e_p$ a total ordering on $E(G)$, define

$$\partial_d[G,\omega] = \sum_{i=0}^{p} (-1)^i \big[G\backslash e_i, \omega|_{E(G)\backslash\{e_i\}}\big], \qquad \partial_c[G,\omega] = \sum_{i=0}^{p} (-1)^i \big[G/e_i, \omega|_{E(G)\backslash\{e_i\}}\big]$$

where $\omega|_{E(G)\backslash\{e_i\}}$ is the total order induced from ω by omitting e_i.

Lemma 5.19. *The map*

$$C_* \to C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_*|;\mathbb{Q})$$
$$[G,\omega] \mapsto f_*^{G,\omega}(\iota_n)$$

is a chain map.

Proof. The boundary of $(\Delta^1)^n$ is the union of the 2^n many embeddings

$$(\Delta^1)^{n-1} \xleftarrow{\approx} (\Delta^1)^j \times \Delta^0 \times (\Delta^1)^{n-j-1} \xrightarrow{1\times d^i \times 1} (\Delta^1)^n \tag{70}$$

for $i \in \{0,1\}$ and $j \in \{0,\dots,n-1\}$. Writing $\partial_{i,j} : (\Delta^1)^{n-1} \to (\Delta^1)^n$ for this embedding and writing $\iota_n \in C_n^{\mathrm{sing}}((\Delta^1)^n;\mathbb{Q})$ for the fundamental cycle (68), the Leibniz rule for cross product leads to the convenient formula

$$\begin{aligned}\partial\iota_n &= \sum_{i=1}^{n} (-1)^{i-1} \iota_{i-1} \times (\partial\iota_1) \times \iota_{n-i} \in C_{n-1}(\partial(\Delta^1)^n;\mathbb{Z}) \\ &= \sum_{i=1}^{n} (-1)^{i-1} (\partial_{1,j})_* \iota_{n-1} - \sum_{i=1}^{n} (-1)^{i-1} (\partial_{0,j})_* \iota_{n-1}.\end{aligned} \tag{71}$$

The lemma then follows because the composition

$$(\Delta^1)^{n-1} \xrightarrow{\partial_{i,j}} (\Delta^1)^n \xrightarrow{f^{G,\omega}} |\mathcal{F}_\bullet|$$

is $f^{G',\omega'}$, where G' is the graph obtained from G by either collapsing (for $i=0$) or deleting (for $i=1$) the jth edge in G, and ω' is the restriction of the total order ω to $E(G') = E(G)\backslash\{e_j\}$. □

In order to obtain the graph spectral sequence, we must consider $|\overline{\mathcal{F}}_\bullet|$ with its filtration by the subspaces

$$F_n|\overline{\mathcal{F}}_\bullet| = |F_n\overline{\mathcal{F}}_\bullet| \subset |\overline{\mathcal{F}}_\bullet|$$

where $F_n\overline{\mathcal{F}}_p$ corresponds to isomorphism classes of triangular diagrams in which $b_1(G_{0,p}) \le n$. Fortunately the entire argument for the quasi-isomorphism $(C_*,\partial) \simeq C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|;\mathbb{Q})$ applies essentially verbatim to the simplicial subset $F_n\overline{\mathcal{F}}_\bullet \subset \overline{\mathcal{F}}_\bullet$. We state the conclusion.

Theorem 5.20. *Let $F_nC_* \subset C_*$ be the span of those generators $[G,\omega]$ for which $b_1(G) \le n$. This is a subcomplex for all n, and the formula (69) restricts to quasi-isomorphisms*

$$F_nC_* \xrightarrow{\simeq} C_*^{\mathrm{sing}}(F_n|\overline{\mathcal{F}}_\bullet|;\mathbb{Q})$$

for all n. In other words, the formula (69) defines a filtered quasi-isomorphism $(C_*,\partial) \to C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|;\mathbb{Q})$. □

By this theorem, the graph spectral sequence is isomorphic to the spectral sequence associated to the filtered chain complex (C_*, ∂, F) starting on E^1, at least as a spectral sequence of rational vector spaces. It is rather easy to upgrade this isomorphism to one of spectral sequences of algebras, where (C_*, ∂, F) is upgraded to a filtered DGA by declaring $[G, \omega][G', \omega'] = [G \sqcup G', \omega * \omega']$, where $\omega * \omega'$ denotes the "concatenation" order of $E(G \sqcup G') = E(G) \sqcup E(G')$ where all edges of G come before all edges of G'. The coproduct is rather more involved.

Theorem 5.21. *Let (C_*, ∂) have the coproduct induced by*

$$\Delta([G, \omega]) = \sum_{\gamma \subset E(G)} [G_{|\gamma}, \omega_{|\gamma}] \otimes [G/\gamma, \omega_{|E(G) \setminus \gamma}] \tag{72}$$

where $V(G_{|\gamma}) = V(G)$ and $E(G_{|\gamma}) = \gamma$ while G/γ is the graph obtained by collapsing each element of γ to a point. The edge sets of $G_{|\gamma}$ and G/γ are in canonical bijection with γ and $E(G) \setminus \gamma$ respectively, and $\omega_{|\gamma}$ and $\omega_{|E(G)\setminus\gamma}$ denote the restriction of the total order ω to these subsets.

1. *This coproduct makes (C_*, ∂, F) into a filtered coalgebra.*
2. *The map (69) is compatible with the map induced by the space level map $|\overline{\mathcal{F}}_\bullet| \approx |\mathrm{es}\overline{\mathcal{F}}_\bullet| \to |\overline{\mathcal{F}}_\bullet| \times |\overline{\mathcal{F}}_\bullet|$: the diagram*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\Delta"] & C_*(|\overline{\mathcal{F}}_\bullet|; \mathbb{Q}) \arrow[d] \\ C \otimes C \arrow[r] & C_*(|\overline{\mathcal{F}}_\bullet| \times |\overline{\mathcal{F}}_\bullet|; \mathbb{Q}), \end{tikzcd} \tag{73}$$

commutes up to a filtration-preserving chain homotopy, where the lower horizontal map is the composition of (69)⊗(69) and the cross product.

Proof sketch. It is easy to see that $b_1(G_{|\gamma}) + b_1(G/\gamma) = b_1(G)$.

The second statement is more involved, and the diagram (73) does not commute on the nose. The starting point for the homotopy is the cycle $\iota_1' = \iota_1 + \partial c \in C_1(\Delta^1; \mathbb{Z})$ where $c \in C_2(\Delta^1; \mathbb{Z})$ is the simplex

$$\Delta^2 \to \Delta^1$$
$$(t_0, t_1, t_2) \mapsto t_0(1,0) + t_1\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + t_2(0,1).$$

In other words, ι_1' is the sum (in the abelian group $C_1(\Delta^1; \mathbb{Z})$) of the two maps

$$\Delta^1 \to \Delta^1$$
$$(t_0, t_1) \mapsto \left(\tfrac{1}{2}t_0, 1 - \tfrac{1}{2}t_0\right)$$
$$(t_0, t_1) \mapsto \left(1 - \tfrac{1}{2}t_1, \tfrac{1}{2}t_1\right),$$

precisely the two non-degenerate 1-simplices in the edgewise subdivision of $\Delta^1_\bullet$. It follows that the composition

$$C_1(|\Delta^1_\bullet|) \xrightarrow{\cong} C_1(|\mathrm{es}(\Delta^1_\bullet)|) \to C_1(|\Delta^1_\bullet| \times |\Delta^1_\bullet|)$$

sends $\iota_1' \mapsto d^0 \times \iota_1 + \iota_1 \times d^1$.

Then

$$\iota_n' = \iota_1' \times \cdots \times \iota_1' \in C_n((\Delta^1)^n; \mathbb{Z})$$

gives a different set of chain level representatives of the fundamental class of the cube $(\Delta^1)^n$, with the exact same formal properties as the ι_n. This includes the formula (71), which holds with ι'_n and ι'_{n-1} in place of ι_n and ι_{n-1}.

The formula $\iota'_n = (\partial c) \times \iota'_{n-1} + \iota \times \iota'_{n-1}$ implies inductively that $\iota'_n = \iota_n + \partial c_n$ for

$$c_n = \sum_{j=1}^{n} \iota_{j-1} \times c \times \iota'_{n-j}.$$

Then the two chain maps

$$\begin{aligned} C &\to C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|; \mathbb{Q}) \\ [G,\omega] &\mapsto f_*^{G,\omega}(\iota_n) \\ [G,\omega] &\mapsto f_*^{G,\omega}(\iota'_n) \end{aligned}$$

are chain homotopic. One then checks that the diagram (73) commutes strictly, if the top horizontal map is replaced by the chain homotopic map $[G,\omega] \mapsto f_*^{G,\omega}(\iota'_n)$. □

Corollary 5.22. *Let* $\mathrm{Gr}(C)$ *denote the associated graded of the filtered chain complex* $C = (C_*, \partial)$*, with bigrading in which a generator* $[G,\omega]$ *has bidegree* $(b_1(G), |E(G)| - b_1(G))$*. The differential has bidegree* $(0,-1)$*.*

The map (69) *induces an isomorphism of bigraded bialgebras* $H_*(\mathrm{Gr}(C)) \to {}^G E^1_{*,*}$ *where the coproduct on* $H_*(\mathrm{Gr}(C))$ *is induced by the Connes–Kreimer formula* (72)*.*

Each generator $[G,\omega] \in C$ represents a non-zero element of $\mathrm{Gr}_g(C)$ for $g = b_1(G)$ which we denote by the same notation $[G,\omega]$. Then $\mathrm{Gr}(C)$ inherits a product and coproduct from C, making it a bigraded bialgebra (isomorphic to C as a bigraded bialgebra). The boundary map $\partial = \partial_d + \partial_c : C \to C$ induces a boundary map on $\mathrm{Gr}(C)$ given by a similar formula on generators $[G,\omega] \in \mathrm{Gr}(C)$ but omitting the terms involving $[G',\omega']$ where $b_1(G') < b_1(G)$. Precisely, this amounts to dropping the terms in $\partial_c([G,\omega])$ in which a loop is collapsed, and omitting the terms in $\partial_d([G,\omega])$ in which a bridge is deleted.

Proof. We have already shown that (69) is a filtered quasi-isomorphism, so it induces an isomorphism on homology. The filtered homotopy in the theorem implies that the isomorphism on homology is a map of coalgebras. That it is also a map of algebras is similar but easier, we omit the details. □

5.4.1. *E^1 page for the primed and double-primed variants.* By similar arguments, we obtain graph complexes C'_* and C''_* and a diagram of comparison maps

$$\begin{tikzcd} C_* \arrow[r] \arrow[d, "\simeq"] & C'_* \arrow[r] \arrow[d, "\simeq"] & C''_* \arrow[d, "\simeq"] \\ C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}_\bullet|; \mathbb{Q}) \arrow[r] & C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}'_\bullet|; \mathbb{Q}) \arrow[r] & C_*^{\mathrm{sing}}(|\overline{\mathcal{F}}''_\bullet|; \mathbb{Q}), \end{tikzcd}$$

where the vertical maps are DGA maps and filtered quasi-isomorphisms, and compatible with the coproduct up to filtration-preserving chain homotopy. Here, C'_* is a graph complex with generators $[G,\omega]$ where all path components of G have positive genus, and C''_* is a graph complex with generators $[G,\omega]$ where all path components of G have positive genus and G

furthermore have no bridges or cut vertices. It is important to realize these complexes C'_* and C''_* as quotients of (C_*, ∂) rather than subcomplexes. We spell out the precise definition.

Definition 5.23. *Let*

$$C'_* = C_*/I'$$
$$C''_* = C_*/I''$$

where $I'_* \subset C_*$ is the ideal generated by the elements $([p]-1)$ and $[T,\omega]$ where p denotes the graph with one vertex and no edges and T runs over all trees with at least one edge. In other words, it is the $\mathbb{Q}$-linear subspace generated by the elements $[T \sqcup G, \omega]$ and $[p \sqcup G, \omega] - [G,\omega]$ as G runs over all graphs and T runs over all tree with at least one edge.

$I''_* \subset C_*$ is linearly generated by the subspace I'_* together with $[G,\omega]$ and $[G_1 \vee G_2, \omega] - [G_1 \sqcup G_2, \omega]$ as G runs over all graphs with at least one bridge and (G_1, v_1) and (G_2, v_2) run over graphs with at least one edge and chosen vertices $v_i \in V(G_i)$. Here $G_1 \vee G_2$ denotes the graph obtained from $G_1 \sqcup G_2$ by gluing v_1 to v_2.

Lemma 5.24. *The product on C_* descends to products on C'_* and C''_*, i.e. I' and I'' are ideals. Moreover the coproduct on C_* descends to coproducts on C'_* and C''_*, given by the same formula as in Theorem 5.21.*

Proof sketch. This may either be checked by hand, or deduced from the space-level structure: for instance, $C_* \to C''_*$ is the map induced on cellular chains from the map of filtered spaces $|\overline{\mathcal{F}}_\bullet| \to |\overline{\mathcal{F}}''_\bullet|$, which preserves the space-level product and coproduct. □

We have explained why the spectral sequence ${}^G E^r_{*,*}$ comes with a product and coproduct, satisfying a Leibniz rule on each page, and it is easy to see that there are unit and counit morphisms $\mathbb{Q} \to {}^G E^1_{0,0} \to \mathbb{Q}$ induced by filtered maps $\{*\} \to |\overline{\mathcal{F}}_\bullet| \to \{*\}$, making this a spectral sequence of bialgebras. As it stands, the E^1 page does not admit an antipode though.

Lemma 5.25. *The space $F_0|\overline{\mathcal{F}}_\bullet|$ consists of two path components, both of which have trivial rational homology. Consequently, ${}^G E^1_{0,0} \cong \mathbb{Q}^2$ and ${}^G E^1_{0,t} = 0$ for $t > 0$.*

Proof. We may instead calculate the homology of the chain complex $F_0 C_* \subset C_*$ whose generators $[G,\omega]$ have $b_1(G) = 0$. In other words, G is a forest. $H_0(F_0 C_*)$ may be calculated by hand: if we write p^n for the graph with no edges and $n \in \mathbb{N}$ many vertices, then these form a basis for $F_0 C_0$. There is a unique isomorphism class of 1-edge graphs with n many vertices and $b_1 = 0$ for each $n \geq 2$, and its boundary is $p^n - p^{n-1}$. In terms of $F_0|\overline{\mathcal{F}}_\bullet|$, this shows that the space is the disjoint union of precisely two path components, one corresponding to the empty graph and one corresponding to all non-empty graphs.

To see that $H_*(F_0 C_*) = 0$ for $* > 0$ we consider the homomorphism $T : F_0 C_n \to F_0 C_{n+1}$ given by

$$T([G,\omega]) = \sum_{G = G_1 \vee G_2} [G_1 \vee e \vee G_2], \tag{74}$$

where the sum is indexed by all possible ways of writing $G = G_1 \vee G_2$ as the wedge sum of two graphs G_1 and G_2 at basepoints $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, with $E(G_1) \neq \emptyset \neq E(G_2)$. The graph $G_1 \vee e \vee G_2$ is obtained from the disjoint union $G_1 \sqcup G_2$ by inserting a new edge e

connecting v_1 to v_2, and $E(G_1 \vee e \vee G_2) = \{e\} \sqcup E(G_1) \sqcup E(G_2)$ is ordered by declaring e less than any element of $E(G_1)$, all of which are declared less than any element of $e(G_2)$. Then T is a chain homotopy from the identity to the chain map $S : F_0C_* \to F_0C_*$ defined by

$$S([G, \omega]) = (\partial \circ T - T \circ \partial - \mathrm{id})([G, \omega]).$$

If G is a forest all of whose trees have at most n edges for $n \geq 2$, then $S([G, \omega])$ is a linear combination of forests all of whose trees have at most $n-1$ edges. In other words, S is a chain homotopy from the identity map to a chain map which strictly decreases the maximal number of edges in trees for any tree with more than 1 edge. We now consider a map of CDGAs

$$\mathbb{Q}[p, e] \to F_0C_*$$

whose domain is the free graded-commutative algebra on a generator p of degree 0 and a generator e of degree 1, with boundary defined by $\partial p = 0$ and $\partial i = p - p^2$. The map defined by sending p to a one-vertex graph and e to a graph with two vertices connected by an edge. Iterating the chain homotopy S shows that any cycle in F_0C_* is homologous to a cycle in the image of this map of CDGAs. Finally, an easy calculation shows that $H_j(\mathbb{Q}[p, e], \partial) = 0$ for $j > 0$. See also [Wil15, Proposition 3.4] for a related result. ☐

Corollary 5.26. *Let $x \in {}^GE^1_{0,0} = H_0(F_0|\overline{\mathcal{F}}_\bullet|; \mathbb{Q})$ be the path component corresponding of the non-empty graphs. Then ${}^GE^\infty_{0,0} \cong \mathbb{Q}[x]/(x^2 - x)$ as a ring, with coproduct given by $\Delta(x) = x \otimes x$ and counit by $x \mapsto 1$.*

As already explained, this bialgebra represents the functor $R \mapsto \{x \in R \mid x^2 = x\}$, which is a monoid scheme but not a group scheme, so the bialgebra does not admit an antipode.

Proof. We have already seen that the ring structure is as stated, with $x = [p]$ in the notation of the above proof. The equation $\Delta(x) = x \otimes x$ follows from the fact that the coproduct on the E^∞ page is induced by the space-level diagonal map. Similarly for the counit. ☐

We can now prove that the primed version of the spectral sequence is a localization of the unprimed version.

Proposition 5.27. *The class $x \in E^1_{0,0}$ survives to a class $x \in E^r_{0,0}$ for all $r \leq \infty$. The map of spectral sequences ${}^GE^r_{*,*} \to {}^{G'}E^r_{*,*}$ sends $x \mapsto 1$ and factors through an isomorphism*

$$\mathbb{Q}[x^{\pm 1}] \otimes_{\mathbb{Q}[x]} {}^GE^r_{*,*} \xrightarrow{\cong} {}^{G'}E^r_{*,*}.$$

Proof. The class survives for degree reasons, and $(x-1) \in {}^GE^1_{0,0}$ is in the kernel of the map to ${}^{G'}E^1_{0,0}$ because $N_1\mathcal{F}'_0$ contains a morphism from the empty graph to the point graph so that these become identified in $|\overline{\mathcal{F}}'_\bullet|$ which is connected.

More generally, we then observe that since we defined $\mathcal{F}'_p$ by adding more morphisms, there are canonical surjections

$$F_n\overline{\mathcal{F}}_p \twoheadrightarrow F_n\overline{\mathcal{F}}'_p$$

for all p, so $\overline{\mathcal{F}}'_p$ is a quotient of $\overline{\mathcal{F}}_p$ by an equivalence relation. In fact it is the quotient by a monoid action: the monoidal structure on $\mathcal{F}_p$ given by disjoint union descends to a commutative monoid structure on $\overline{\mathcal{F}}_p$, the subset $F_0\overline{\mathcal{F}}_p \subset \overline{\mathcal{F}}_p$ is a submonoid, and $\overline{\mathcal{F}}'_p$ can be identified with the projection onto the quotient monoid, i.e. the coequalizer

$$(F_0\overline{\mathcal{F}}_p) \times (F_n\overline{\mathcal{F}}_p) \rightrightarrows F_n\overline{\mathcal{F}}_p$$

of the action map and the projection map. Moreover, each equivalence class under this monoid action has a canonical representative in $\overline{\mathcal{F}}_p$, given by removing all genus-zero components of $G_{0,p}$. More specifically, regarded as a set with $F_0\overline{\mathcal{F}}_p$-action the set $F_n\overline{\mathcal{F}}_p$ is free on the subset of graphs without genus-zero components. Taking free $\mathbb{Q}$-modules, this implies that $\mathbb{Q}\langle F_n\overline{\mathcal{F}}_p\rangle$ is a free $\mathbb{Q}\langle F_0\overline{\mathcal{F}}_p\rangle$-module for all n and that we have an isomorphism of vector spaces

$$\mathbb{Q}\langle F_n\overline{\mathcal{F}}_p\rangle \otimes_{\mathbb{Q}\langle F_0\overline{\mathcal{F}}_p\rangle} \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}\langle F_n\overline{\mathcal{F}}'_p\rangle. \tag{75}$$

Freeness of the monoid action implies that (75) models the derived tensor product, and we obtain a spectral sequence

$$\mathrm{Tor}^{H_*(|F_0\overline{\mathcal{F}}_\bullet|)}(H_*(F_n|\overline{\mathcal{F}}_\bullet|), \mathbb{Q}) \Rightarrow H_*(|F_n\overline{\mathcal{F}}'_\bullet|).$$

The Proposition now follows from the ring isomorphism $H_*(|F_0\overline{\mathcal{F}}_\bullet|) \cong \mathbb{Q}[x]/(x^2-x)$, that $\mathbb{Q} = \mathbb{Q}[x]/(x-1)$ is projective as a $\mathbb{Q}[x]/(x^2-x)$-module, that $\mathbb{Q}[x]/(x-1) \cong (\mathbb{Q}[x]/(x^2-x))[x^{-1}]$, and that tensoring with a projective module is an exact functor so that the base change formula propagates through the spectral sequence. □

As explained above, C''_* is filtered quasi-isomorphic to $C_*(|\overline{\mathcal{F}}''_\bullet|;\mathbb{Q})$ for the same reason as C_* is filtered quasi-isomorphic to $C_*(|\overline{\mathcal{F}}_\bullet|;\mathbb{Q})$, namely the (rational) cell structure of $|\overline{\mathcal{F}}''_\bullet|$. For all three variants, the boundary map $\partial = \partial_c + \partial_d$ will in general have terms corresponding to edge collapse and to edge deletion. When G is a graph without bridges, then $b_1(G') < b_1(G)$ for any graph G' arising from G by deleting an edge. In the chain complexes $\mathrm{Gr}(C)$, $\mathrm{Gr}(C')$, and $\mathrm{Gr}(C'')$ we therefore have $\partial([G,\omega]) = \partial_c([G,\omega])$ when G has no bridges. At this point we see the main advantage of the double-primed variant over the other two: the edge-deletion map $\partial_d : C'' \to C''$ strictly decreases filtration and hence induces the zero map

$$\partial_d = 0 : \mathrm{Gr}(C''_*) \to \mathrm{Gr}(C''_*)$$

on associated graded. Therefore only contributions from ∂_c remains in $\partial : \mathrm{Gr}(C''_*) \to \mathrm{Gr}(C''_*)$ and we arrive at the familiar formula

$$\begin{aligned} \mathrm{Gr}(C''_*) &\xrightarrow{\partial} \mathrm{Gr}(C''_*) \\ [G,\omega] &\mapsto \sum_{e\in E(G)} \pm[G/e, \omega|_{E(G)\setminus\{e\}}], \end{aligned} \tag{76}$$

where the sum is over all non-loop edges of G, the sign is $(-1)^{\omega(e)}$ if $\omega : E(G) \to \{0,\ldots,p-1\}$ is an order-preserving bijection.

The formula (76) for the induced boundary map on associated graded is the main justification for the specific definition of $\mathcal{F}''_\bullet$ in Definition 5.4. In both chain complexes $\mathrm{Gr}(C_*)$ and $\mathrm{Gr}(C'_*)$ some remnant of ∂_d will remain in the boundary map, namely a (signed) sum of bridge deletion. Absence of bridge-deletion terms allow us to establish the following precise relationship between the E^1 page of the double-primed version of the graph spectral sequence and the cochain complex GC_2 considered by Kontsevich, Willwacher, and others.

Proposition 5.28. *Let*

$$L^\vee = \mathrm{Indec}(\mathrm{Gr}(C''_*))$$

be the graph complex obtained as the indecomposables in the differential graded Hopf algebra $\mathrm{Gr}(C''_*)$: explicitly, the subcomplex of $\mathrm{Gr}(C''_*)$ spanned by non-empty graphs, quotiented by the subcomplex spanned by graphs with more than one path component.

This chain complex $L^\vee$ is generated by symbols $[G,\omega]$ where G is a non-empty connected graph with $b_1(G) > 0$ and without bridges or cut vertices (no valence conditions), and with boundary map defined as alternating sum of edge contractions. It also inherits a Lie cobracket by anti-symmetrizing the Connes–Kreimer coproduct. If we write $L = \mathrm{Prim}(\mathrm{Gr}(C''_*)^\vee)$ for the linear dual, then L inherits a DG Lie algebra structure. Passing to homology, we deduce an isomorphism of bigraded Lie coalgebras

$$\mathrm{Indec}\big({}^{G''}E^1_{*,*}\big) \cong H_*(L^\vee)$$

and a dual isomorphism of bigraded Lie algebras

$$\mathrm{Prim}\big({}^{G''}E_1^{*,*}\big) \cong H^*(L).$$

Moreover, there is a split extension of Lie algebras

$$H^*(\mathsf{GC}_2) \hookrightarrow H^*(L) \twoheadrightarrow L^2$$

where L^2 denotes the bigraded vector space which is $\mathbb{Q}$ in bidegrees $(1,4k)$ for all $k \geq 0$ and is 0 in all other bidegrees, equipped with the trivial Lie algebra structure.

Proof. We have already constructed a filtered quasi-isomorphism between C''_* and the singular chains of $|N_\bullet \mathcal{F}''_\bullet|$, giving an isomorphism

$${}^{G''}E^1_{*,*} \cong H_*(\mathrm{Gr}(C''_*)).$$

Since rational homology commutes with forming indecomposables for commutative Hopf algebras, we deduce an isomorphism of bigraded Lie coalgebras

$$\mathrm{Indec}\big({}^{G''}E^1_{*,*}\big) \cong H_*(L^\vee).$$

and a dual statement about primitives in the dual Hopf algebras.

Next we relate the DG Lie algebra L to the graph complex GC_2. The former permits 2-valent vertices while the latter is generated by graphs with vertices of valence at least 3, so we start by studying summands of L given by valency conditions. The following argument is essentially [Wil15, Proposition 3.4]. The only changes are that we apply it to the "one-vertex-irreducible" case directly, and that for completeness, we spell out some of the details that were sketched in op. cit.. We observe that

$$L \cong L^2 \oplus L^{2,3} \oplus L^3$$

where L^2, $L^{2,3}$, and L^3 are the subcomplexes of L generated by graphs with

- all vertices of valence 2;
- at least one vertex of valence 2 and at least one vertex of valence ≥ 3; and
- all vertices of valence ≥ 3,

respectively. Let us argue that these are subcomplexes. The generators of L^2 are cycles C_ℓ of length ℓ, in bidegree $(1, \ell-1)$, and C_ℓ is nonzero if and only if $\ell = 4k+1$ for some $k \geq 0$. It is thus clear that L^2 is a subcomplex, supported in bidegrees $(1, 4k)$ for $k \geq 0$. It is also clear that $L^{2,3}$ is a subcomplex, since the property of having a vertex of valence 2, respectively at least 3, is preserved under uncontraction of edges. Finally, L^3 is also a subcomplex. Indeed, suppose $[G, \omega]$ is a nonzero generator of L^3, and suppose G' is a 1-edge uncontraction of G with a 2-valent vertex v, incident to edges e and f. Necessarily $e \neq f$ or else $[G', \omega] = 0$. Now if G' has any 1-edge contractions in L^3 then they must be G/e and G/f. But these graphs are isomorphic, and in fact $[G'/e, \omega_{|G'/e}] + [G'/f, \omega_{|G'/f}] = 0$. Therefore $\partial[G, \omega] \in L^3$.

We now verify that $L^{2,3}$ is acyclic. For notational convenience, in this proof of acyclicity, we will regard $L^{2,3}$ merely as a singly graded chain complex, graded by number of edges, with differential in degree -1. Filter $L^{2,3}$ by the number of ≥ 3-valent vertices: for each $q > 0$, let $F^q L^{2,3} \subset L^{2,3}$ be the subcomplex generated by graphs with at most q vertices of valence at least 3. Note that $F^0 = 0$. Furthermore, while each inclusion $F^q L^{2,3} \subset L^{2,3}$ is strict, note that for each number g, for q sufficiently large—specifically, $q \geq 2g-2$—the subcomplex $F^q L^{2,3}$ contains the direct summand of $L^{2,3}$ of graphs with fixed first Betti number g. So it is enough to prove that F^q/F^{q+1} is acyclic. For any generator $[G, \omega]$ of F^q/F^{q+1}, let the *stabilization* of G, temporarily denoted $\underline{G}$, be the graph obtained from G by "smoothing" all vertices of valence 2. Each graph G can naturally be recorded as $\underline{G}$ together with positive integers on each edge of $\underline{G}$, and the boundary of $[G, \omega]$ in the quotient F^q/F^{q+1} is a formal sum of graphs $\underline{G}$ obtained by changing one of those positive integers from k to $k+1$. This corresponds to an alternating sum of $k+1$ terms in the differential, which is 0 if and only if k is odd. Thus, let T denote the cochain complex

$$0 \to \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\cong} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\cong} \cdots$$

where the leftmost $\mathbb{Q}$ appears in degree 1. Then

$$F^q L^{2,3}/F^{q+1} L^{2,3} \cong \bigoplus_G \big(T^{\otimes e_G}/\mathbb{Q}[-e_G]\big)/\mathrm{Aut}(G),$$

where the direct sum is over representatives for the isomorphism classes of graphs G with exactly q vertices and minimum valence at least 3, and $e_G = |E(G)|$ denotes the number of edges of G. The quotient by $\mathbb{Q}[-e_G]$ corresponds to the fact that the stabilized graphs themselves do not appear as nonzero generators in $L^{2,3}$, since they have no vertices of valence 2.

But $T^{\otimes e_G}$ is quasi-isomorphic to $\mathbb{Q}[-e_G]$, so $T^{\otimes e_G}/\mathbb{Q}[-e_G]$ and $(T^{\otimes e_G}/\mathbb{Q}[-e_G])/\mathrm{Aut}(G)$ are acyclic, and so too is $F^q L^{2,3}/F^{q+1} L^{2,3}$. Finally, L^3 is isomorphic to the "one-vertex-irreducible" graph complex GC_{1vi} [Wil15]. As shown in [Wil15] and in [CGV05], the inclusion $\mathsf{GC}_{1vi} \hookrightarrow \mathsf{GC}_2$ is a quasi-isomorphism, so we have

$$L \xrightarrow{\simeq} L^2 \oplus \mathsf{GC}_{1vi} \overset{\simeq}{\hookrightarrow} L^2 \oplus \mathsf{GC}_2.$$

The isomorphism $H^*(L^3) \cong H^*(\mathsf{GC}_2)$ is a Lie algebra isomorphism by comparing definitions. For degree reasons, this additive splitting makes

$$H^*(L^3) \hookrightarrow H^*(L) \twoheadrightarrow H^*(L^2)$$

into a split extension of Lie algebras, and $L^2 \cong H^*(L^2)$ must have trivial Lie bracket. $\square$

The map of filtered spaces $|\overline{\mathcal{F}}''_\bullet| \simeq_{\mathbb{Q}} |N_\bullet \mathcal{F}''_\bullet| \to BK(\mathbb{Z})$ induces a map of spectral sequences of Hopf algebras, of the form

$$ {}^{G''}E^r_{*,*} \to {}^{Q}E^r_{*,*} \tag{77} $$

abutting to the induced map

$$ H_*(|\overline{\mathcal{F}}''_\bullet|) \to H_*(BK(\mathbb{Z})). $$

When passing to primitives in the linear duals of the respective E^1 pages, it therefore induces a map of Lie algebras

$$ \mathrm{Prim}\big({}^{Q}E_1^{*,*}\big) \to \mathrm{Prim}\big({}^{G''}E_1^{*,*}\big) \cong H^*(L) $$

from the primitives in the E^1 page of the (cohomological) Quillen spectral sequence to the semidirect product. Bidegrees are such that primitives in ${}^{Q}E_1^{s,t}$ are mapped to graphs with first Betti number s and $s+t$ many edges. Restricted to diagonal bidegrees, this can be identified with a map of Lie algebras

$$ \mathrm{Prim}\Big(\bigoplus_g {}^{Q}E_1^{g,g}\Big) \to \mathrm{Prim}\Big(\bigoplus_g {}^{G''}E_1^{g,g}\Big) \cong H^0(\mathsf{GC}_2) \cong \mathfrak{grt}_1, \tag{78} $$

with the last isomorphism following from [Wil15, Theorem 1.1].

5.5. **Comparison to other graph complexes and spectral sequences.** Let us briefly comment on the relationship between our graph complexes $C \to C' \to C''$ and other graph complexes appearing in the literature. The closest is the graph complex denoted fGC by Willwacher and others, since that is also built out of all graphs with no conditions on connectivity or the valency of vertices, and where "tadpoles" are allowed. Reading the definition closely though, one notices that the graphs in fGC must be non-empty, which ours need not be. By comparing definitions we therefore obtain an isomorphism of cochain complexes

$$ C^\vee \cong \mathsf{fGC} \oplus \mathbb{Q}.\emptyset, $$

where $\emptyset$ denotes the graph with no vertices and no edges. It is of some importance for us to include the empty graph, for our Hopf algebra to be unital and co-unital.

If we let $T \subset C$ be the DG ideal spanned by graphs containing at least one edge from a vertex to itself (i.e., a graph containing a tadpole), then the filtered cochain complex

$$ (\mathrm{Indec}(C/T))^\vee $$

is a filtered DG Lie algebra isomorphic to the one used in [KWŽ17] for producing their spectral sequence. It follows that there is a map of spectral sequences from theirs to the linear dual of the indecomposables of (the unprimed version of) ours.

6. Freeness of a Lie algebra generated by ω^{4k+1} classes

We now prove that the images of the elements $\omega^5, \omega^9, \dots, \omega^{45}$ generate a free Lie subalgebra in $\mathrm{Prim}(W_0 H_c^*(\mathcal{A}) \otimes \mathbb{R})$. We will make use of the map (78) from the primitives in the Quillen spectral sequence (along diagonal bidegrees) to $H^0(\mathsf{GC}_2)$ constructed in the previous section, as well as the following results from the literature:

(1) Willwacher's theorem [Wil15, Theorem 1.1] which gives an isomorphism of graded Lie algebras between $H^0(\mathsf{GC}_2)$ and the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$.

(2) An injective map [Bro12] from the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}} \to \mathfrak{grt}$, where $\mathfrak{g}^{\mathfrak{m}}$ is isomorphic to the free graded Lie algebra on certain generators σ_{2k+1} in every degree $2k+1 \geq 3$. These generators σ_{2k+1} are canonical modulo Lie words in generators σ_{2j+1} with $j < k$.
(3) The integration pairing between $\omega^{2g-1} \in {}^{Q}E_1^{g,g} \otimes \mathbb{R}$ (see (23)) and the locally-finite homology class of the wheel $[W_g]$ is non-zero for $g > 1$ odd [BS24]. In other words, the image of ω^{2g-1} in $H^0(\mathsf{GC}_2) \otimes \mathbb{R}$, under the map (78), pairs non-trivially with $[W_g] \in H_0(\mathsf{GC}_2^{\vee})$.

This last point (3) relies on the discussion of subsection 2.5, in the following way. The differential form ω^{2g-1} most naturally gives a class in the E_1 page of the cohomological *tropical spectral sequence*, while the wheel graph W_g gives a class in the homological *Quillen spectral sequence*. In order to make sense of the pairing we use the explicit comparison discussed in §2.5, which we recall involved a zig-zag

$$\mathrm{Gr}_g(BK(\mathbb{Z})) \xleftarrow{\simeq} |N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \xrightarrow{\simeq_{\mathbb{Q}}} (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}.$$

We use this to make sense of pairing a reduced homology class in $\mathrm{Gr}_g(BK(\mathbb{Z}))$ with a reduced cohomology class in $(P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}$, or equivalently a compactly supported cohomology class in $P_g/\mathrm{GL}_g(\mathbb{Z})$. To implement the comparison, we will first factor the second arrow through a space $|\overline{T}_\bullet(\mathbb{Q}^g) \cup \{\infty\}|$ which we define now. Let $\overline{T}_p(\mathbb{Q}^g)$ be the quotient of the set $N_0T_p(\mathbb{Q}^g)$ by the equivalence relation that $(A_p \subset \cdots \subset A_0, <) \sim (A'_p \subset \cdots \subset A'_0, <')$ provided the resulting maps

$$\Delta^p \setminus \partial\Delta^p \to P_g,$$

given by the formula (28) are in the same orbit for the action of $\mathrm{GL}_g(\mathbb{Z})$-action on the set of such maps. We point out that the formula (28) for the map associated to $a = (A_p \subset \cdots \subset A_0, <)$ does not involve the total order $<$ at all, and that the elements $\psi \in A_0$ enter the formula only through their squares $v \mapsto (\psi(v))^2$. Therefore the element of $\overline{T}_p(\mathbb{Q}^g)$ represented by a is invariant under any change of $<$, under replacing some ψ by $-\psi$, and by definition also under precomposing all ψ's with the same $X \in \mathrm{GL}_g(\mathbb{Z})$.

The definition of the equivalence relation defining $\overline{T}_p(\mathbb{Q}^g)$ ensures that the map from Proposition 2.21 factors as

$$\mathrm{Gr}_g(BK(\mathbb{Z})) \xleftarrow{\simeq} |N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to |\overline{T}_\bullet(\mathbb{Q}^g) \cup \{\infty\}| \to (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}.$$

We may also define a map of simplicial sets

$$\mathrm{Gr}_g(\mathcal{F}''_\bullet) \xrightarrow{f} \overline{T}_\bullet(\mathbb{Q}^g) \cup \{\infty\}$$

by sending the isomorphism class of a flag $x = (G_{0,0} \hookrightarrow \cdots \hookrightarrow G_{0,p}) \in F_g\mathcal{F}''_p$ to the element of $T_p(\mathbb{Q}^g) \cup \{\infty\}$ given as $f(x) = \infty$ if $b_1(G_{0,p}) < g$ and otherwise by first choosing an isomorphism $H_1(G_{0,p}; \mathbb{Z}) \cong \mathbb{Z}^g$ and then letting

$$f(x) = [(A_p \subset \cdots \subset A_0, <)] \in \overline{T}_p(\mathbb{Q}^g),$$

where $A_0 \subset H^1(G_{0,p}; \mathbb{Q}) \cong (\mathbb{Q}^g)^\vee$ is the subset defined by first choosing a section of $H(G_{0,p}) \twoheadrightarrow E(G_{0,p})$ and then letting A_0 be the image of the resulting composition

$$E(G_{0,p}) \hookrightarrow H(G_{0,p}) \hookrightarrow C^1(G; \mathbb{Q}) \twoheadrightarrow H^1(G; \mathbb{Q}) \cong \mathbb{Q}^g.$$

Let $A_i \subset A_0$ be the image of the subset $E(G_{0,p-i}) \subset E(G_{0,p})$ for $i = 0, \ldots, p$ and choose an arbitrary total ordering $<$ of A_0 to define the element $f(x)$. (The resulting element $f(x)$ does not depend on the choice of $<$, nor the choice of representative half-edges of each edge, by definition of the equivalence relation defining $\overline{T}_\bullet(\mathbb{Q}^g)$.) These maps fit into a homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_g\,|N_\bullet \mathcal{F}''_\bullet| \arrow[r, "H_1"] \arrow[dd, "\simeq_{\mathbb{Q}}"] & \mathrm{Gr}_g(BK(\mathbb{Z})) & \\
& {|N_\bullet T_\bullet(\mathbb{Q}^g) \cup \{\infty\}|} \arrow[u, "\simeq"'] \arrow[d] \arrow[dr, "\simeq_{\mathbb{Q}}"] & \\
\mathrm{Gr}_g\,|\overline{\mathcal{F}}''_\bullet| \arrow[r, "f"] & {|\overline{T}_\bullet(\mathbb{Q}^g) \cup \{\infty\}|} \arrow[r] & (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}.
\end{tikzcd}$$

Now, a bridgeless graph G with $b_1(G) = g$ defines a map

$$(\Delta^1_\bullet)^{E(G)} \to \mathrm{Gr}_g(\overline{\mathcal{F}}_\bullet)$$

whose geometric realization is one of the cells in the rational cell structure leading to the cellular chain complex $\mathrm{Gr}_g(C'')$ calculating the reduced homology of $\mathrm{Gr}_g(|\overline{\mathcal{F}}''_\bullet|)$ which is the column ${}^{G''}E^1_{g,*}$ in the double-primed graph spectral sequence. Composing with the map above, we obtain a map

$$(\Delta^1)^{E(G)} \to (P_g/\mathrm{GL}_g(\mathbb{Z})) \cup \{\infty\}.$$

For $g = 2k+1$, integrating the differential form ω^{4k+1} along this map for each G with $|E(G)| = 4k+2$ defines a element of degree $4k+2$ in the linear dual cochain complex

$$\omega^{4k+1} \in \mathrm{Gr}_g(C'')^\vee = \mathrm{Hom}(\mathrm{Gr}_g(C''), \mathbb{Q}).$$

This element is a cocycle and pairs with the cycle $[W_{2k+1}] \in \mathrm{Gr}_g(C'')$ by evaluating the integral from [BS24], and by the discussion in §2.5 this is the desired pairing.

6.1. **Proof of Theorem 1.3.**

Lemma 6.1. *For all $g > 1$ odd, the wheel classes $[W_g] \in {}^{G''}E^1_{g,g}$ are primitive, and hence annihilated by the Lie cobracket. Consequently, the wheel classes in $H_0(\mathsf{GC}_2^\vee)$ define linear maps*

$$[W_g] : \mathfrak{grt} \to \mathbb{Q}$$

which vanish on commutators $[\mathfrak{grt}, \mathfrak{grt}]$, and hence descend to maps $\mathfrak{grt}^{\mathrm{ab}} \to \mathbb{Q}$, which are nonzero by (3).

Proof. The coproduct on ${}^{G''}E^1$ is induced by (72). Since every proper nonempty subgraph of a wheel W_g has a vertex of degree < 3 (or, if one prefers, since every non-trivial contraction of a number of edges in a wheel graph produces a doubled edge), it follows that any term $\gamma \otimes W_g/\gamma$ in the reduced coproduct is zero in $\mathsf{GC}_2^\vee \otimes \mathsf{GC}_2^\vee$. It follows that $\Delta[W_g] = 1 \otimes [W_g] + [W_g] \otimes 1$. As a result, the Lie cobracket on $H_*(L^\vee)$, which is obtained by antisymmetrizing Δ, vanishes on $[W_g]$. $\square$

The fact that the wheel class $[W_{2k+1}] \in H_0(\mathsf{GC}_2^\vee)$ is non-zero in $\mathrm{Hom}(\mathfrak{grt}^{\mathrm{ab}}, \mathbb{Q})$ may also be deduced from a theorem of Rossi and Willwacher [RW14], asserting that it pairs non-trivially with the image of σ_{2k+1} in $\mathfrak{grt} \cong H^0(\mathsf{GC}_2)$ for every $k > 1$.

Proposition 6.2. *For $g > 1$ odd, the image of the canonical class $[\omega^{2g-1}] \in {}^Q E_1^{g,g} \otimes_{\mathbb{Q}} \mathbb{R}$ under the map (78) has non-zero image in the abelianisation $\mathfrak{grt}^{\mathrm{ab}} \otimes_{\mathbb{Q}} \mathbb{R}$.*

Proof. Let $g > 1$ and suppose on the contrary that $[\omega^{2g-1}] \in [\mathfrak{grt}, \mathfrak{grt}] \otimes_{\mathbb{Q}} \mathbb{R}$ lies in the subspace of commutators of $H^0(\mathsf{GC}_2) \otimes_{\mathbb{Q}} \mathbb{R}$. Then by the previous lemma, the pairing $\langle [\omega^{2g-1}], [W_g] \rangle$ must vanish, contradicting item (3). □

We immediately deduce a number of consequences. The weight filtration on $\mathfrak{g}^{\mathfrak{m}}$ and $\mathfrak{grt}$ is induced by a grading which we denote by W. For the former, this is defined to be (minus) half the Hodge-theoretic weight. For the latter, $\mathfrak{grt}$ is by definition embedded as a vector space in the free graded Lie algebra on two generators $\mathbb{L}(X, Y)$ where X, Y are assigned weight 1. This weight has the property that any generator $\sigma_{2k+1} \in \mathfrak{g}^{\mathfrak{m}}$ has weight $2k+1$, and so that the weight coincides with the grading by the genus (or equivalently by one half of the cohomological degree) on the diagonal $\bigoplus_g {}^G E_1^{g,g}$.

Corollary 6.3. *Let $N > 0$ odd such that $\mathrm{Gr}_k^W \mathfrak{g}^{\mathfrak{m}} = \mathrm{Gr}_k^W \mathfrak{grt}$ for $k \leq N$. Then the images of*

$$\omega^5, \ldots, \omega^{2N-1}$$

generate a free Lie algebra in $\mathfrak{grt} \otimes_{\mathbb{Q}} \mathbb{R}$. Consequently

$$T\left(\bigoplus_{k=1}^{\frac{N-1}{2}} \omega^{4k+1}[-1]\mathbb{Q}\right) \to \bigoplus_{g \geq 0} {}^Q E_1^{g,g} \otimes_{\mathbb{Q}} \mathbb{R}$$

is injective.

Proof. We have established that for all $g > 1$ odd, the image of $[\omega^{2g-1}]$ in $\mathfrak{grt} \otimes \mathbb{R}$ is non-trivial in $\mathfrak{grt}^{\mathrm{ab}} \otimes \mathbb{R}$ and has graded weight g. If $g \leq N$ then the image of $[\omega^{2g-1}]$ lies in the graded Lie subalgebra $\mathfrak{g}^{\mathfrak{m}} \otimes \mathbb{R} \subset \mathfrak{grt} \otimes \mathbb{R}$ by the assumption, and must have non-zero image in $(\mathfrak{g}^{\mathfrak{m}})^{\mathrm{ab}} \otimes \mathbb{R}$. By [Bro12], $\mathfrak{g}^{\mathfrak{m}}$ is a free Lie algebra, with generators given by any homogeneous choice of representatives for $(\mathfrak{g}^{\mathfrak{m}})^{\mathrm{ab}} \otimes \mathbb{R} = H_1(\mathfrak{g}^{\mathfrak{m}}; \mathbb{R})$ in $\mathfrak{g}^{\mathfrak{m}}$. It follows that $\omega^5, \ldots, \omega^{2N-1}$ generate a free graded Lie algebra $\mathfrak{g}'$ inside $\mathfrak{grt} \otimes \mathbb{R}$, and hence inside $\mathrm{Prim}({}^Q E_1)$. The Milnor–Moore theorem then implies that the universal enveloping algebra $\mathcal{U}\mathfrak{g}'$ embeds into $\bigoplus {}^Q E_1^{g,g} \otimes_{\mathbb{Q}} \mathbb{R}$. The last statement follows from the fact that the universal enveloping algebra of a free graded Lie algebra is isomorphic to the free tensor algebra on its generators.

Establishing Corollary 6.3 for a given N reduces to a finite, but possibly very large, computation. In practice, it is equivalent to show the equality of finite-dimensional vector spaces $W_N \mathcal{O}(\mathfrak{g}^{\mathfrak{m}}) = W_N \mathcal{O}(\mathfrak{grt})$ which can be verified by interpreting elements of their affine rings as formal symbols (corresponding to multiple zeta values) modulo relations and checking that their dimensions agree. Indeed, extensive computer calculations of the latter imply that the assumption of the corollary holds for $N = 23$ (see, for example, [BBV10, p. 2], where it was asserted that the dimension was computed up to weight 24). □

One could possibly push this further using more recent results and techniques which exploit the Lie algebra structure. In any case, this corollary may be combined with previous results to produce a huge amount of cohomology: we deduce that the symmetric algebra on $\Omega_c^*[-1]$ and the non-trivial commutators in $\omega^5, \ldots, \omega^{45}$ embeds into the E_1 page of the cohomological Quillen spectral sequence.

6.2. **Depth filtration.** The de Rham fundamental Lie algebra of the projective line minus 3 points ([Del89], see also [Bro21a] and the references therein) is canonically isomorphic to the free graded Lie algebra $\mathbb{L}(X, Y)$ on two generators X, Y (corresponding to generators of $H^1_{dR}(\mathbb{P}^1 \backslash \{0, 1, \infty\}; \mathbb{Q}) \cong \mathbb{Q}X \oplus \mathbb{Q}Y$). The depth filtration on $\mathbb{L}(X, Y)$ is the decreasing filtration such that elements of depth r are linear combinations of Lie brackets involving at least r occurrences of the letter Y. It induces a decreasing depth filtration D on both the Grothendieck–Teichmüller and motivic Lie algebras. It is known that

$$\mathrm{Gr}^1_D\, \mathfrak{g}^{\mathfrak{m}} = \mathrm{Gr}^1_D\, \mathfrak{grt} \;,$$

and furthermore that $D^1\mathfrak{grt} = \mathfrak{grt}$, and $[\mathfrak{grt}, \mathfrak{grt}] \subset D^2\mathfrak{grt}$. It follows that there is a natural surjective map $\mathfrak{grt}^{\mathrm{ab}} \to \mathrm{Gr}^1_D\, \mathfrak{grt}$. A question of Drinfeld's, which asks if $\mathfrak{g}^{\mathfrak{m}} \to \mathfrak{grt}$ is surjective, would imply that $\mathfrak{grt}^{\mathrm{ab}} \to \mathrm{Gr}^1_D\, \mathfrak{grt}$ is an isomorphism (it is known that $\mathrm{Gr}^1_D\, \mathfrak{g}^{\mathfrak{m}} = (\mathfrak{g}^{\mathfrak{m}})^{\mathrm{ab}}$ is an isomorphism). We have the following stronger version of Proposition 6.2.

Proposition 6.4. *For $g > 1$ odd, the image of the forms $[\omega^{2g-1}]$ are non-zero in $\mathrm{Gr}^1_D\, \mathfrak{grt} \otimes_{\mathbb{Q}} \mathbb{R}$.*

Proof. The proof of [Wil15, Proposition 9.1] shows that the isomorphism $\phi : H^0(\mathsf{GC}_2) \to \mathfrak{grt}$ constructed in *loc. cit.* lifts to a map from GC_2 to the free Lie algebra on two generators $\mathbb{L}(X, Y)$, and furthermore has the property that the only graph whose image involves the Lie word $\mathrm{ad}(X)^{2n}(Y)$ is the wheel W_{2n+1}. It follows that the image $\phi(G)$ for all connected graphs G not isomorphic to a wheel lies in $D^2\mathbb{L}(X, Y)$, since the depth filtration is induced by the degree in Y, and hence $\mathrm{Gr}^1_D\, \mathbb{L}(X, Y)$ is generated in weight $2n + 1$ by precisely $\mathrm{ad}(X)^{2n}(Y)$. Denote the increasing filtration on $H_0(GC)$ dual to the depth filtration on $\mathfrak{grt}$ by $D_\bullet$. By the above, it has the property that $D_1H_0(GC)$ is spanned by the wheel classes $[W_{2n+1}]$ for $n > 1$. Since $[\omega^{2g-1}]$ pairs non-trivially with the wheel $[W_g]$, for $g > 1$ odd, it follows that $[\omega^{2g-1}] \in \mathfrak{grt} \otimes_{\mathbb{Q}} \mathbb{R}$ is not contained in D^2. $\square$

Unfortunately, the bigraded Lie algebra generated by $\mathrm{Gr}^1_D\, \mathfrak{grt} = \mathrm{Gr}^1_D\, \mathfrak{g}^{\mathfrak{m}}$ is not free, and has quadratic relations coming from cusp forms (Ihara–Takao). It is nevertheless very large.

Corollary 6.5. *The image of the Lie algebra generated by $\{\omega^5, \omega^9, \ldots\} \subset \mathrm{Prim}(W_0H^*_c(\mathcal{A}; \mathbb{R}))$ surjects onto the bigraded Lie subalgebra of $\mathrm{Gr}^*_D\, \mathfrak{g}^{\mathfrak{m}}$ generated by $\mathrm{Gr}^1_D\, \mathfrak{g}^{\mathfrak{m}}$ which has one generator in every odd degree > 1, namely the images of the σ_{2k+1} modulo D^2.*

For example, one can deduce from this result and [Zag93], that

$$\dim_{\mathbb{R}} \bigoplus_{k+\ell=N} [\omega^{4k+1}, \omega^{4\ell+1}]\mathbb{R} \geq \left\lfloor \frac{N}{3} \right\rfloor \;. \tag{79}$$

This demonstrates the highly non-commutative nature of the product on ${}^{Q}E_1^{*,*}$, going beyond the range of degrees covered by Corollary 6.3.

7. Further results and conjectures

7.1. Symmetric products of canonical forms and an announcement of Ronnie Lee. An immediate corollary of Theorem 4.5 is:

Corollary 7.1. *The map* (55) *induces an embedding of bigraded vector spaces*

$$\mathrm{Sym}\,(\Omega_c^*[-1]) \hookrightarrow W_0 H_c^*(\mathcal{A};\mathbb{R})$$

where $\mathrm{Sym}\,(\Omega_c^*[-1]) \subset T(\Omega_c^*[-1])$ *is the vector subspace of (graded-commutative) symmetrised products. By applying the isomorphism* $W_0 H_c^*(\mathcal{A};\mathbb{Q}) \otimes \mathbb{Q}[x]/x^2 \cong {}^{Q}E_1^{*,*}$*, with* x *in bidegree* $(1,0)$*, we deduce an embedding of bigraded vector spaces:*

$$\mathrm{Sym}\,(\Omega_c^*[-1]) \otimes \mathbb{Q}[x]/x^2 \hookrightarrow \bigoplus_{g,d} H_c^{d-g}(P_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R}) \ . \tag{80}$$

Proof. Since $W_0 H_c^*(\mathcal{A};\mathbb{Q})$ is connected, cocommutative, and finite dimensional in each degree, the Milnor–Moore theorem gives an isomorphism $\mathcal{U}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{Q}))) \xrightarrow{\sim} W_0 H_c^*(\mathcal{A};\mathbb{Q})$ of bigraded Hopf algebras. The Poincaré–Birkhoff–Witt theorem implies that the natural map $\mathrm{Sym}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{Q}))) \to \mathrm{Gr}\,\mathcal{U}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{Q})))$ is an isomorphism of algebras, where Gr is the grading associated to the filtration induced by length. In particular, the map given by symmetrisation of products $\mathrm{Sym}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{Q}))) \to \mathcal{U}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{Q})))$ is an isomorphism of vector spaces. Thus we deduce an isomorphism of bigraded vector spaces:

$$\mathrm{Sym}(\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{R}))) \xrightarrow{\sim} W_0 H_c^*(\mathcal{A};\mathbb{R})$$

By Theorem 1.2, $\Omega_c^*[-1]$ embeds into $\mathrm{Prim}(W_0 H_c^*(\mathcal{A};\mathbb{R}))$.

The second statement follows from the description of the E_1 page of the cohomological Quillen spectral sequence in terms of compactly supported cohomology

$${}^{Q}E_1^{s,t} \cong H_t(\mathrm{GL}_s(\mathbb{Z}); \mathrm{St}_s \otimes \mathbb{Q})^\vee \cong H_c^{s+t}(P_s/\mathrm{GL}_s(\mathbb{Z});\mathbb{Q}),$$

see (5) and (8).

□

A version of the following injective map, which is implied by (80),

$$\mathrm{Sym}\,(\Omega_c^*[-1]) \hookrightarrow \bigoplus_g H_c^*(P_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R})$$

was announced by Ronnie Lee in 1978 [Lee78], but no proof has ever appeared in print. Note that (80) gives rise to infinitely many new classes in the cohomology of the groups $\mathrm{SL}_g(\mathbb{Z})$.

Example 7.2. By Corollary 7.1, we have the following nonzero classes in $H_c^*(P_6/\mathrm{GL}_6(\mathbb{Z});\mathbb{R})$:

$$[\omega^9].\epsilon \ , \qquad [\omega^5].[\omega^5] = [\omega^5|\omega^5] \ , \qquad [\omega^5 \wedge \omega^9].\epsilon \ ,$$

in degrees $11, 12, 16$ respectively. In $H_c^*(P_7/\mathrm{GL}_7(\mathbb{Z});\mathbb{R})$ we obtain:

$$[\omega^5|\omega^5].\epsilon \ , \quad [\omega^{13}] \ , \quad [\omega^5 \wedge \omega^{13}] \ , \quad [\omega^9 \wedge \omega^{13}] \ , \quad [\omega^5 \wedge \omega^9 \wedge \omega^{13}] \ ,$$

in degrees $13, 14, 19, 23, 28$. Here, a dot denotes the symmetrised product: $a.b = \frac{1}{2}(a \times b + (-1)^{\deg a \cdot \deg b} b \times a)$ where $\times$ is the multiplication in ${}^{Q}E_1$. It follows from the computations of [EVGS13] that these are the only non-vanishing classes in the range for which the compactly supported cohomology of $P_g/\mathrm{GL}_g(\mathbb{Z})$ has been computed in its entirety.

Remark 7.3. A much stronger version of the Corollary 7.1 holds. By a similar application of the Milnor–Moore theorem, the symmetric algebra generated by:

(1) Independent elements in the Lie algebra generated by $\{\omega^5, \omega^9, \ldots\}$ inside $\mathrm{Prim}\,{}^{Q}E_1 \otimes \mathbb{R}$
(2) The homogeneous elements in $\Omega_c^*[-1]$ of the form $\omega^{4i_1+1} \wedge \ldots \wedge \omega^{4i_k+1}$ for $k > 1$
(3) The generator ϵ in bidegree $(1, 0)$
(4) Infinitely many elements of the form (84) (see below)

embeds as a bigraded vector space into ${}^{Q}E_1 \otimes \mathbb{R}$, because these elements are primitives. The Lie algebra in (1) is at least as large as: the free Lie algebra on $\{\omega^5, \ldots, \omega^{45}\}$, and, the part of the depth-graded motivic Lie algebra generated in depth 1.

7.2. **Poincaré series and dimensions.**

Let

$$P(s,t) = \sum_{g,n \geq 0} \dim \left(T(\Omega_c^*[-1])_{g,n}\right) \, s^g t^n$$

denote the Poincaré series of the bigraded vector space $T(\Omega_c^*[-1])$ with respect to genus and degree minus genus. Let us define for all $k \geq 1$

$$f_{2k+1}(t) = t^{4k+2} \prod_{i=1}^{k-1} (1 + t^{4i+1}) \ .$$

Then, for example, we have

$$f_3(t) = t^6 \ , \ f_5(t) = (1 + t^5)t^{10} \ , \ f_7(t) = (1 + t^5)(1 + t^9)t^{14} \ .$$

Each polynomial $f_{2k+1}(t)$ is the Poincaré series for the graded vector space $\Omega_c^*(2k+1)[-1]$. Since $\Omega_c^*[-1]$ is the direct sum of all $\Omega_c^*(2k+1)[-1]$ it follows that

$$P(s,t) = \frac{1}{1 - \sum_{k \geq 1} f_{2k+1}(t) s^{2k+1}} = 1 + s^3 t^6 + s^5(t^{10} + t^{15}) + t^{12} s^6 + \ldots$$

Since $f_{2k+1}(-1) = 0$ for all $k > 1$, an interesting consequence is that the generating function for the Euler characteristic is $P(s, -1) = (1 - s^3)^{-1}$. It follows that the Euler characteristic (with respect to degree) of the genus g component of $T(\Omega_c^*[-1])$ is congruent to 1 if $g \equiv 0 \pmod 3$ and 0 otherwise.

7.3. **Diagonals and degrees.**

We now refine Corollaries 1.4, 1.5, and 1.9. The Poincaré series for the tensor algebra $T(\bigoplus_{k \geq 1} \omega^{4k+1}\mathbb{Q})$ generated by the classes ω^{2g-1} in bidegree (g, g) which features in Question 1.16 is

$$P(s,t) = \frac{1}{1 - s^3 t^6 - s^5 t^{10} - \ldots} = \frac{1 - s^2 t^4}{1 - s^2 t^4 - s^3 t^6}$$

The coefficient of $s^n t^{2n}$ in $P(t)$ is asymptotically α^{-n} where $\alpha = .7548\cdots$ is the real root of $s^3 + s^2 - 1 = 0$. Corollary 6.3 and the comments which follow prove that the asymptotic growth of the diagonal $\bigoplus_g W_0 H_c^{2g}(\mathcal{A}_g; \mathbb{Q})$ is eventually bounded below by $\alpha_{\mathrm{approx}}^{-n}$ where $\alpha_{\mathrm{approx}} = 0.7551\cdots$ is the real root of $s^{23} + s^{21} + \ldots + s^3 - 1 = 0$, and is therefore very close to what we would expect if Question 1.16 were true.

By multiplying by any symmetric tensor generated by the elements of $\Omega_c^*[-1]$ with the form $\omega^{4i_1+1} \wedge \ldots \wedge \omega^{4i_k+1}$ where $k > 1$, we obtain an additional copy of the diagonal Lie algebra generated by the $\{\omega^{4k+1}, k > 1\}$ which, by the computation above, has exponential growth (see Remark 7.3). Here follow two further applications.

7.3.1. *Refinement of Corollary 1.5.* The dimension of $W_0 H_c^{2g+k}(\mathcal{A}_g)$ grows at least exponentially with g for all non-negative integers k except possibly k in the set

$$S_{\mathcal{A}} = \{1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 15, 16, 19, 20, 23, 24, 28, 32, 36, 40\}.$$

Indeed, by Corollary 1.4, it suffices to show that, for each $k \notin S_{\mathcal{A}}$, there is a genus g such that the bigraded vector space $\Omega_c^*[-1]$ is nonzero in genus g and degree $2g + k$.

Each homogeneous basis element is of the form

$$\omega = \omega^{4k_1+1} \wedge \cdots \wedge \omega^{4k_r+1},$$

where $k_1 < \ldots < k_r$. Then ω is in genus $g = 2k_r + 1$ and degree $(4k_1 + 1) \cdots + (4k_{r-1} + 1)$. The claim follows, since the numbers that can be written as a sum of distinct integers that are at least 5 and congruent to $1 \mod 4$ are

$$5, 9, 13, 14, 17, 18, 21, 22, 25, 26, 27, 29, 30, 31, 33, 34, 35, 37, 38, 39,$$

and all integers greater than or equal to 41.

7.3.2. *Refinement of Corollary 1.9.* The dimension of $H^{\binom{n}{2}-n-k}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$ grows at least exponentially for all integers $k \geq -1$ except possibly k in the set

$$S_{\mathrm{SL}} = \{1, 2, 3, 6, 7, 10, 11, 15, 19, 23\}.$$

We have already noted the bigraded isomorphism

$$(W_0 H_c^*(\mathcal{A}))^\vee \otimes_{\mathbb{Q}} \mathbb{Q}[x]/x^2 \cong \bigoplus_g H_*(\mathrm{GL}_g(\mathbb{Z}), \mathrm{St}_g \otimes \mathbb{Q}),$$

where x has genus 1 and degree 1. Likewise, we have noted that $H^{\binom{n}{2}-k}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$ contains $H_k(\mathrm{GL}_n(\mathbb{Z}), \mathrm{St}_n \otimes \mathbb{Q})$ as a summand. Thus, the dimension of $H^{\binom{n}{2}-n-k}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$ grows exponentially with n unless both k and $k + 1$ are in $S_{\mathcal{A}}$.

7.4. **Polynomials in the wheel homology classes.** The cohomology classes defined above are only defined over the real numbers, because their pairing with rational homology cycles are given by period integrals which include odd values of the zeta function. However, it was shown in [BS24] that the wheel classes give explicit rational homology classes: for all odd $g > 1$ there exists an explicit non-zero locally finite homology class:

$$[\tau_{W_g}] \in H_{2g}^{BM}(P_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{Q}) \ . \tag{81}$$

A corollary of the existence of the Hopf algebra structure on the E^1 page of the homological Quillen spectral sequence implies the following:

Theorem 7.4. *There is an injective map of commutative bigraded algebras*

$$\mathbb{Q}[W_3, W_5, \ldots, W_g, \ldots] \otimes \mathbb{Q}[x]/x^2 \longrightarrow {}^Q E^1 \tag{82}$$

where W_g denotes the wheel class (81) in odd genus g and degree $2g$, and where the element x is in degree 1 and genus 1 and maps to e. The bigrading on the left hand side is by genus and degree minus genus.

Proof. By their definition, the wheel classes (81) factor through the graph spectral sequence ${}^{G}E^1 \to {}^{Q}E^1$. They are primitive in ${}^{Q}E^1$ because this is true a fortiori in the graph complex, by Lemma 6.1. The element x, which represents a one-vertex, one-edge loop in ${}^{G}E^1_{1,0}$, is primitive for reasons of degree. The Milnor–Moore theorem, which holds for any primitively-generated connected graded Hopf algebra of finite type, implies that $\mathcal{U}(\mathrm{Prim}({}^{Q}E^1))$ injects into ${}^{Q}E^1$. By Poincaré–Birkhoff–Witt, the universal enveloping algebra generated by these classes is precisely $\mathbb{Q}[W_3, W_5, \ldots] \otimes \mathbb{Q}[x]/x^2$. □

It was shown in [BS24] that the wheel classes W_g pair non-trivially with the primitive canonical forms ω^{2g-1}. It follows that the dual to (82), tensored with $\mathbb{R}$, is the map

$$W_0 H^*_c(\mathcal{A};\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{R}[\omega^5, \omega^9, \ldots, \omega^{2g-1}, \ldots] \ .$$

which sends all other primitives to zero. In fact, one may replace $\mathbb{R}$ in the previous map with the $\mathbb{Q}$-algebra generated by odd zeta values $\zeta(2n+1)$, for $n \geq 1$.

7.5. **A spectral sequence on $T(\Omega^*_c[-1])$.** Recall the map

$$T(\Omega^*_c[-1] \otimes \mathbb{R}) \to W_0 H^*_c(\mathcal{A};\mathbb{R})$$

which is injective on $\Omega^*_c[-1]$ by Theorem 4.5. Here we construct a spectral sequence whose E_1 page is $T(\Omega^*_c[-1])$ and compare it to the cohomological Quillen spectral sequence. Consider the graded exterior algebra $\wedge P^*$ on the graded vector space $P^* = \bigoplus_{k\geq 1} \mathbb{Q}\beta^{4k+1}$ with generators β^{4k+1} in degree $4k+1$. Endowed with the zero differential, $\wedge P^*$ defines a connected differential graded algebra. The bar construction $B(\wedge P^*)$ is a graded commutative Hopf algebra over $\mathbb{Q}$ generated by symbols $[p_1|\ldots|p_n]$ in degree $\deg(p_1) + \ldots + \deg(p_n) + n$ where $\deg(p_i) > 0$ (note that a more standard convention is to have a minus before the n; this is not the case here), where p_i are homogeneous generators of $\wedge P^*$, and is isomorphic as a vector space to the tensor algebra $T(\wedge(P^*_+))$, where P^*_+ denotes the part of P^* in positive degree. In addition it is equipped with the (graded-commutative) signed shuffle product, and the deconcatenation coproduct

$$\Delta[p_1|\ldots|p_n] = \sum_{i=1}^{n} [p_1|\ldots|p_i] \otimes [p_{i+1}|\ldots|p_n] \ .$$

In addition, there is an internal differential

$$\begin{aligned} d_I : B(\wedge P^*) \quad &\longrightarrow \quad B(\wedge P^*) \\ d_I([p_1|\ldots|p_n]) \quad &= \quad \sum_{i=1}^{n-1} (-1)^i [sp_1|\ldots|sp_{i-1}|sp_i \wedge p_{i+1}|p_{i+2}|\ldots|p_n] \end{aligned} \tag{83}$$

where $s : \wedge P^* \to \wedge P^*$ is the linear map of graded vector spaces which multiplies by $(-1)^n$ in degree n. The differential d_I has degree -1 (owing to the plus sign in our convention for the degree of bar elements) and satisfies $d_I^2 = 0$. It is compatible with the Hopf algebra structures.

Define an increasing filtration G on $\wedge P^*$ as follows: $G_g(\wedge P^*)$ is spanned by elements $\omega^{4i_1+1} \wedge \ldots \wedge \omega^{4i_k+1}$ such that $i_1 < \ldots < i_k$ and $4i_k + 1 \leq 2g - 1$. It defines a filtration $G(\wedge P^*)$ of differential graded algebras. In fact, it follows from the definition that $G_g \wedge G_h \subset G_{\max\{g,h\}}$. The filtration G induces a filtration on $B(\wedge P^*)$, which we also denote by G. It is a filtration of graded Hopf algebras, which is respected by the differential d_I.

Proposition 7.5. *The filtered complex $(B(\wedge P^*), d_I)$ with filtration G defines a spectral sequence ${}^{c}E^r_{s,t}$ of commutative bigraded Hopf algebras such that*

$$ {}^{c}E^1_{s,t} = \mathrm{Gr}^G_s B_{s+t}(\wedge P^*) \ . $$

This spectral sequence converges to the bigraded Hopf algebra $\mathrm{Gr}^G \mathrm{Sym}(P^*_+[-1])$ which is isomorphic to the polynomial ring in primitive generators β^{4k+1} in degree $4k+2$ and genus $2k+1$. Furthermore, the differentials d_1, d_2 vanish, and $d_r[\beta^{4k+1}] = 0$ for all k, r.

Proof. The spectral sequence ${}^{c}E^1_{s,t}$ defined by the filtration G on $(B(\wedge P^*), d_I)$ has E^1 page isomorphic to

$$ {}^{c}E^1_{s,t} = H_{s+t}(\mathrm{Gr}^G_s B(\wedge P^*)) $$

Since the differential d_I strictly decreases the genus, it is identically zero on the associated graded of $B(\wedge P^*)$, and hence $H_{s+t}(\mathrm{Gr}^G_s B(\wedge P^*)) = \mathrm{Gr}^G_s B_{s+t}(\wedge P^*)$. It follows from the Koszul duality between the symmetric and exterior algebras that there is an isomorphism of graded commutative Hopf algebras

$$ H_* \left(B(\wedge P^*), d_I\right) \cong \mathrm{Sym}(P^*_+[-1]) \ . $$

The spectral sequence therefore converges to $\mathrm{Gr}^G H(B(\wedge P^*), d_I) \cong \mathrm{Gr}^G \mathrm{Sym}(P^*_+[-1])$, as bigraded Hopf algebras. The fact that the differential d_r annihilates β^{4k+1} is clear from the definition of d_I, which acts trivially on $[\beta^{4k+1}]$. The fact that d_1, d_2 vanish follows from the definition of d_I, and the fact that the map $[\beta^{4a+1}|\beta^{4b+1}] \mapsto \beta^{4a+1} \wedge \beta^{4b+1}$ sends a term of genus $2(a+b)+2$ to one of genus $2\max\{a,b\}+2$, and therefore decreases the genus by at least 3. $\square$

By identifying P^* with the graded dual of Ω^*_c, we may interpret $B(\wedge P^*)$ as the graded dual of the tensor Hopf algebra $T(\Omega^*_c[-1])$. The proposition therefore defines by duality a spectral sequence on $T(\Omega^*_c[-1])$, which we call the *canonical spectral sequence*, denoted by ${}^{c}E^{s,t}_r$. The differentials in this spectral sequence vanish on elements $[\omega^{4k+1}]$ and, for $r = 2\min\{a,b\}+1$, send $[\omega^{4a+1} \wedge \omega^{4b+1}]$ to the commutator $[\omega^{4a+1}, \omega^{4b+1}] = [\omega^{4a+1}|\omega^{4b+1}] - [\omega^{4b+1}|\omega^{4a+1}]$.

We expect that the map in Theorem 4.5 may be promoted (possibly after rescaling the action of the differentials) to a map of spectral sequences

$$ {}^{c}E^{*,*}_r \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow {}^{Q}E^{*,*}_r \otimes_{\mathbb{Q}} \mathbb{R} $$

which induces an isomorphism on their abutments (which are formally isomorphic, by the previous proposition). This provides yet more evidence of a different kind for Conjecture 1.15.

7.6. **Illustration.** Table 7.6 depicts $T(\Omega^*_c[-1])$. The entries that are known to be isomorphic to $W_0H^*_c(\mathcal{A};\mathbb{Q})$ are highlighted, in particular for $g \leq 7$. Blank entries vanish for dimension reasons. Entries in **low genus** follow from computer calculations of [EVGS13] and [DSEVKM19].

There are two infinite ranges in which the cohomology of $W_0H^*_c(\mathcal{A};\mathbb{Q})$ has been completely determined. The **zero** entries in the bottom three rows follow from [Gun00, BPS23, BMP+22], which imply that

$$ W_0H^{g+n}_c(\mathcal{A}_g;\mathbb{Q}) = 0 \quad \text{ for } 0 \leq n \leq 2, g \geq 1 \ . $$

The bottom two rows also follow from [LS76] and [CP17]. A conjecture of Church-Farb-Putnam implies that $W_0H^n_c(\mathcal{A}_g;\mathbb{Q})$ vanishes for $n < 2g$ (below the diagonal line $n = 2g$). The entries in **high degrees** follow from [Bro23, §14.5], [BBC+24] and imply that for all $g > 1$ odd

$$W_0 H_c^n(\mathcal{A}_g;\mathbb{Q}) \cong \Omega_c^n(g)[-1] \quad \text{and} \quad W_0 H_c^{n-1}(\mathcal{A}_{g-1};\mathbb{Q}) = 0$$

for $n \geq d_g - \kappa(g)$, where d_g is the dimension of $\mathcal{A}_g$ and $\kappa(g)$ is the stable range for the cohomology of the general linear group (which is currently known to be $\kappa(g) \leq g-1$ by [LS19]).

7.6.1. *Application of the Quillen spectral sequence.* Studying the Quillen spectral sequence further allows us to deduce some nonvanishing classes in $W_0H^*(\mathcal{A};\mathbb{R})$ and verifying Conjecture 1.15 up to $g = 9$. Recall that the cohomological Quillen spectral sequence abuts to a free polynomial algebra with generators in bidegree $(2k+1, 2k+1)$, for positive integers k. By inspection of Table 1, we observe that $[\omega^5]$ and $[\omega^9]$ (and in fact, $[\omega^{13}]$, since a class in genus 7 and degree 14 must appear in the abutment of the Quillen spectral sequence) are annihilated by the differentials ${}^Qd_r$ in the Quillen spectral sequence for all r. Furthermore, we know that the primitive element $[\omega^5,\omega^9] = [\omega^5|\omega^9] - [\omega^9|\omega^5] \in {}^QE_1^{8,8}$ of degree 16, which is non-zero by Corollary 6.3, does not appear in the abutment. Since it is annihilated by all differentials ${}^Qd_r$, it must be in the image of a class of degree 15 in genus ≤ 7. The only such class is $[\omega^5 \wedge \omega^9]$ in genus 5. We conclude that

$${}^Qd_r\left([\omega^5\wedge\omega^9]\right) = \begin{cases} \alpha[\omega^5,\omega^9] & \text{if } r = 3 \\ 0 & \text{else} \end{cases}$$

for some $\alpha \in \mathbb{Q}^\times$. Since $[\omega^5], [\omega^9]$ generate a free Lie algebra in $\mathrm{Prim}\left({}^QE_1\right)$ (Corollary 6.3), we may deduce the non-vanishing of many more elements, and the non-triviality of Lie brackets involving the generator $\omega^5\wedge\omega^9$, which is noteworthy as $\omega^5\wedge\omega^9$ has odd degree.

As a warm-up example, let us first show that $[\omega^5, \omega^5\wedge\omega^9] \in {}^QE_1^{8,13}$ is nonzero. As argued above, both ω^5 and $\omega^5\wedge\omega^9$ represent nonzero classes on ${}^QE_3$, and the Leibniz rule gives ${}^Qd_3[\omega^5,\omega^5\wedge\omega^9] = [\omega^5,[\omega^5,\omega^9]]$, which is nonzero by freeness of the Lie algebra generated by $[\omega^5]$ and $[\omega^9]$. Therefore $[\omega^5,\omega^5\wedge\omega^9] \neq 0$. More generally, we can obtain an infinite supply of linearly independent primitive elements by writing down a Hall basis for the free Lie algebra on two generators $\mathbb{L}(x,y)$, choosing for each one an occurence of $[x,y]$ and replacing it with $\omega^5\wedge\omega^9$, and finally replacing x with ω^5 and y with ω^9. For example:

$$\begin{aligned} (xy)x\,,\ (xy)y \quad &\to \quad [\omega^5\wedge\omega^9,\omega^5]\,,\ [\omega^5\wedge\omega^9,\omega^9] \\ ((xy)x)x\,,\ ((xy)y)x)\,,\ ((xy)y)y \quad &\to \quad [[\omega^5\wedge\omega^9,\omega^5],\omega^5]\,,\ [[\omega^5\wedge\omega^9,\omega^9],\omega^5]\,,\ [[\omega^5\wedge\omega^9,\omega^9],\omega^9] \end{aligned} \tag{84}$$

where, in standard notation (ab) denotes $[a,b]$. In length four the Hall basis

$$((xy)x)x)x\ ,\ ((xy)y)x)x\ ,\ ((xy)y)y)x\ ,\ ((xy)y)y)y\ ,\ ((xy)y)(xy)\ ,\ ((xy)x)(xy)$$

may be lifted to, for example, a set of six elements:

$$[[[\omega^5\wedge\omega^9,\omega^5],\omega^5],\omega^5]\ ,\ [[[\omega^5\wedge\omega^9,\omega^9],\omega^5],\omega^5]\ ,\ [[[\omega^5\wedge\omega^9,\omega^9],\omega^9],\omega^5]$$

$$[[[\omega^5\wedge\omega^9,\omega^9],\omega^9],\omega^9]\ ,\ [[\omega^5,\omega^9],\omega^9],\omega^5\wedge\omega^9]\ ,\ [[\omega^5,\omega^9],\omega^5],\omega^5\wedge\omega^9]$$

They are independent since their images under ${}^Qd_3$ are part of a Hall basis for the free lie algebra $\mathbb{L}(\omega^5,\omega^9)$, and hence are independent. In particular, since the Lie bracket $[\omega^5\wedge\omega^9,\omega^5]$ is non-zero, the Milnor–Moore theorem implies that the two classes in genus 8 and degree 21 of the form $[\omega^5|\omega^5 \wedge \omega^9]$, $[\omega^5 \wedge \omega^9|\omega^5]$ are linearly independent, which proves Conjecture 1.15 up to and including genus 9.

In genus 10, injectivity of the map $T(\Omega_c^*[-1]) \to W_0 H_c^*(\mathcal{A};\mathbb{R})$ in Conjecture 1.15 is not known for four cohomological degrees, namely $d = 34, 30, 29$, and 25. Conjecture 1.15 predicts the existence of $2, 1, 2$, and 4 independent classes respectively, while we prove that the dimension of $W_0 H_c^d(\mathcal{A}_{10};\mathbb{R})$ is at least $1, 0, 1$, and 3 in these degrees, respectively. For example, in degree 30, it is not known if the primitive element $[\omega^5 \wedge \omega^9|\omega^5 \wedge \omega^9]$ is non-zero. See Table 1.

7.7. **Proof of Theorem 1.18.** Following the notation suggested by Namikawa, we shall write $\mathcal{A}_g^{\#}$ for the minimal Satake, or Baily–Borel, compactification of $\mathcal{A}_g$.

Proof. The Satake compactification of $\mathcal{A}_h$ has a natural stratification

$$\mathcal{A}_h^{\#} = \bigsqcup_{g \le h} \mathcal{A}_g.$$

This induces a spectral sequence of mixed Hodge structures ${}^{\#}E_*$ abutting to the cohomology $H^*(\mathcal{A}_h^{\#})$ whose E_1 page is

$${}^{\#}E_1^{g,k} = H_c^{g+k}(\mathcal{A}_g),$$

for $g \le h$, and 0 otherwise. Let us consider the E_1 page of the induced spectral sequence on the weight zero subspaces. Choose $h = 6$, which is sufficiently large so that $H^6(\mathcal{A}_h^{\#})$ is stable. Then ${}^{\#}E_1$ agrees with the truncation of $W_0 H_c^*(\mathcal{A})$ after the $g = 6$ column, which is depicted in Table 1. Similarly, the corresponding spectral sequence abutting to $W_0 H^*(\mathcal{A}_3^{\#})$ is obtained by truncating after the third column. For degree reasons, both of these groups are given by ${}^{\#}E_1^{3,3}$. Thus, we have isomorphisms

$$W_0 H^6(\mathcal{A}_6^{\#}) \cong W_0 H_c^6(\mathcal{A}_3) \cong W_0 H^6(\mathcal{A}_3^{\#}).$$

In particular, the inclusion $\mathcal{A}_3^{\#} \subset \mathcal{A}_6^{\#}$ induces an isomorphism on $W_0 H^6$. Now we show that the induced map on all of H^6 is injective. To see this, note that $H^6(\mathcal{A}_6^{\#})$ has rank 2, with

$$H^6(\mathcal{A}_6^{\#})/W_0 H^6(\mathcal{A}_6^{\#})$$

spanned by the Goresky–Pardon lift $\tilde{c}_3$ of $c_3(\Lambda)$, the third Chern class of the Hodge bundle [CL17, Loo17]. Thus, it will suffice to show that the restriction of $\tilde{c}_3$ to $\mathcal{A}_3^{\#}$ is not zero.

The Goresky–Pardon lift $\tilde{c}_3$ has the following property: fix g and choose a toroidal compactification $\mathcal{A}_g^{\Sigma}$, as in [AMRT75]. Let Λ_g denote the Hodge bundle on $\mathcal{A}_g$ and let $\widetilde{\Lambda}_g$ be the extension to $\mathcal{A}_g^{\Sigma}$ constructed in [Mum77]. Then there is a unique projection $\pi_\Sigma \colon \mathcal{A}_g^{\Sigma} \to \mathcal{A}_g^{\#}$ that extends the identity on the shared open subset $\mathcal{A}_g$, and

$$\pi_\Sigma^*(\tilde{c}_3) = c_3(\widetilde{\Lambda}_g). \tag{85}$$

See [GP02]. We can choose a toroidal compactification so that the Torelli map extends to a morphism from the moduli space of stable curves $\overline{\mathcal{M}}_g \to \mathcal{A}_g^{\Sigma}$ and the pullback of $\tilde{c}_3$ is the Hodge class λ_3. Then $\lambda_3 \neq 0$ in $H^6(\overline{\mathcal{M}}_3)$ because it appears as a multiplicative factor in the integrand of explicit nonzero Hodge integrals [FP00]. It follows that $\tilde{c}_3$ is nonzero in $H^6(\mathcal{A}_3^{\#})$, which proves the claim.

Next, we claim that $\tilde{c}_3$ is in the image of the push-forward for the open inclusion

$$\iota_*\colon H^6_c(\mathcal{A}_3) \to H^6(\mathcal{A}_3^{\dagger\!\dagger}).$$

The property (85) for $g = 2$ shows that the restriction of $\tilde{c}_3$ to $\mathcal{A}_2^{\dagger\!\dagger}$ vanishes, since Λ_2 has rank 2. Applying excision gives a long exact sequence

$$\cdots \to H^6_c(\mathcal{A}_3) \to H^6(\mathcal{A}_3^{\dagger\!\dagger}) \to H^6(\mathcal{A}_2^{\dagger\!\dagger}) \to \cdots$$

Since $\tilde{c}_3$ is in the kernel of the map to $H^6(\mathcal{A}_2^{\dagger\!\dagger})$, it must be in the image of $H^6_c(\mathcal{A}_3)$, as claimed.

We have shown that the rank two stable cohomology group $H^6(\mathcal{A}_6^{\dagger\!\dagger})$ injects into $H^6(\mathcal{A}_3^{\dagger\!\dagger})$ and the image is contained in the image of $H^6_c(\mathcal{A}_3)$, which also has rank 2 [Hai02]. Thus we have a zig-zag of isomorphisms of mixed Hodge structures

$$H^6_c(\mathcal{A}_3) \xrightarrow{\sim} \mathrm{Im}\left(H^6(\mathcal{A}_6^{\dagger\!\dagger}) \to H^6(\mathcal{A}_3^{\dagger\!\dagger})\right) \xleftarrow{\sim} H^6(\mathcal{A}_\infty^{\dagger\!\dagger})$$

Since they come from algebraic maps, they also induce isomorphisms in algebraic de Rham cohomology, and hence remain true in any suitable category of realisations. The main result of [Loo17] shows that $H^6(\mathcal{A}_6^{\dagger\!\dagger})$ is a nontrivial extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}$, whose extension class is given by a nonzero rational multiple of $\zeta(3)$. This means that, in a suitable choice of basis for Betti and de Rham cohomology, the period matrix P is upper-triangular with $\zeta(3)$ above the diagonal, and 1, $(2\pi i)^3$ along the diagonal. Following the zig-zag and applying Poincaré duality to $H^6_c(\mathcal{A}_3)$ shows that $H^6(\mathcal{A}_3)$ is likewise a nontrivial extension of $\mathbb{Q}(-6)$ by $\mathbb{Q}(-3)$. More precisely, its period matrix, with respect to the dual basis, is the inverse transpose of P times $(2\pi i)^6$, and hence lower-triangular with $-\zeta(3)(2\pi i)^3$ below the diagonal. $\square$

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36										$[\boldsymbol{\omega^5 \wedge \omega^9 \wedge \omega^{13} \wedge \omega^{17}}]$	0
35										$\mathbf{0}$	0
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32										$\mathbf{0}$	0
31										$[\boldsymbol{\omega^9 \wedge \omega^{13} \wedge \omega^{17}}]$	0
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27									$\mathbf{0}$	$[\omega^5 \wedge \omega^{13} \wedge \omega^{17}]$	0
26									0	0	0
25									0	0	0
24									0	0	$[\omega^5 \mid \omega^5 \wedge \omega^9 \wedge \omega^{13}]$ $[\omega^5 \wedge \omega^9 \wedge \omega^{13} \mid \omega^5]$
23									0	$[\omega^5 \wedge \omega^9 \wedge \omega^{17}]$	0
22									0	$[\omega^{13} \wedge \omega^{17}]$	0
21								$[\boldsymbol{\omega^5 \wedge \omega^9 \wedge \omega^{13}}]$	0	0	0
20								$\mathbf{0}$	0	0	$[\omega^5 \wedge \omega^9 \mid \omega^5 \wedge \omega^9]$
19								$\mathbf{0}$	0	0	$[\omega^5 \mid \omega^9 \wedge \omega^{13}]$ $[\omega^9 \wedge \omega^{13} \mid \omega^5]$
18								$\mathbf{0}$	0	$[\omega^9 \wedge \omega^{17}]$	0
17								$\mathbf{0}$	0	0	0
16								$[\boldsymbol{\omega^9 \wedge \omega^{13}}]$	0	0	0
15							$\mathbf{0}$	$\mathbf{0}$	0	0	$[\omega^9 \mid \omega^5 \wedge \omega^9]$ $[\omega^5 \wedge \omega^9 \mid \omega^9]$, $[\omega^5 \mid \omega^5 \wedge \omega^{13}]$ $[\omega^5 \wedge \omega^{13} \mid \omega^5]$
14							$\mathbf{0}$	$\mathbf{0}$	0	$[\omega^5 \wedge \omega^{17}]$	0
13							$\mathbf{0}$	$\mathbf{0}$	$[\omega^5 \mid \omega^5 \wedge \omega^9]$ $[\omega^5 \wedge \omega^9 \mid \omega^5]$	0	0
12							$\mathbf{0}$	$[\boldsymbol{\omega^5 \wedge \omega^{13}}]$	0	0	0
11							$\mathbf{0}$	$\mathbf{0}$	0	0	0
10						$[\boldsymbol{\omega^5 \wedge \omega^9}]$	$\mathbf{0}$	$\mathbf{0}$	0	0	$[\omega^9 \mid \omega^9]$ $[\omega^5 \mid \omega^{13}]$, $[\omega^{13} \mid \omega^5]$
9						$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	0	$[\omega^{17}]$ $[\omega^5 \mid \omega^5 \mid \omega^5]$	0
8						$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$[\omega^5 \mid \omega^9]$ $[\omega^9 \mid \omega^5]$	0	0
7						$\mathbf{0}$	$\mathbf{0}$	$[\boldsymbol{\omega^{13}}]$	0	0	0
6					$\mathbf{0}$	$\mathbf{0}$	$[\boldsymbol{\omega^5 \mid \omega^5}]$	$\mathbf{0}$	0	0	0
5					$\mathbf{0}$	$[\boldsymbol{\omega^9}]$	$\mathbf{0}$	$\mathbf{0}$	0	0	0
4					$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	0	0
3				$[\omega^5]$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	0	0
2				$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
1			$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
0	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

TABLE 1. Without the red entries, a table showing generators for a largest known subspace of $W_0 H_c^*(\mathcal{A};\mathbb{R})$ in genus up to 10, deduced from Theorems 1.2 and 1.3, and the discussion in Section 7.6. The (s,t) entry shows linearly independent elements of $W_0 H_c^{s+t}(\mathcal{A}_s;\mathbb{R})$. An entry (s,t) is highlighted in **green**, **blue**, or **purple** if it is known to be all of $W_0 H_c^{s+t}(\mathcal{A}_s;\mathbb{R})$; the color coding is given in Section 7.6. The red entries are additional elements of $T(\Omega_c^*[-1])$ appearing as elements of E_1 of the canonical spectral sequence (Section 7.5), and conjecturally (Conjecture 1.15) new linearly independent generators of $W_0 H_c^*(\mathcal{A};\mathbb{R})$. Thus, the table in its entirety shows ${}^{c}E_1$ and verifies Conjecture 1.15 for all $g \leq 9$.