A TROPICAL APPROACH TO HODGE CONJECTURE FOR POSITIVE CURRENTS

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These are notes for the 2015 Simons Symposium on Tropical and Non-Archimedean Geometry, outlining the joint work of the author with Farhad Babaee [BH15]. Let X be an n-dimensional smooth projective algebraic variety over the complex numbers, and let p and q be nonnegative integers with $p+q=n$. Let us consider the following statements:

(HC) The Hodge conjecture
The Hodge conjecture holds for X in codimension q, that is, the intersection

$$H^{2q}(X,\mathbb{Q}) \cap H^{q,q}(X)$$

consists of classes of p-dimensional algebraic cycles with rational coefficients.

(HC$'$) The Hodge conjecture for currents
If $\mathcal{T}$ is a (p,p)-dimensional real closed current on X with cohomology class

$$\{\mathcal{T}\} \in \mathbb{R} \otimes_{\mathbb{Z}} \big(H^{2q}(X,\mathbb{Z})/\text{tors} \cap H^{q,q}(X)\big),$$

then $\mathcal{T}$ is a weak limit of the form

$$\mathcal{T} = \lim_{i\to\infty} \mathcal{T}_i, \quad \mathcal{T}_i = \sum_j \lambda_{ij}[Z_{ij}],$$

where λ_{ij} are real numbers and Z_{ij} are p-dimensional subvarieties of X.

(HC$^+$) The Hodge conjecture for positive currents
If $\mathcal{T}$ is a (p,p)-dimensional strongly positive closed current on X with cohomology class

$$\{\mathcal{T}\} \in \mathbb{R} \otimes_{\mathbb{Z}} \big(H^{2q}(X,\mathbb{Z})/\text{tors} \cap H^{q,q}(X)\big),$$

then $\mathcal{T}$ is a weak limit of the form

$$\mathcal{T} = \lim_{i\to\infty} \mathcal{T}_i, \quad \mathcal{T}_i = \sum_j \lambda_{ij}[Z_{ij}],$$

where λ_{ij} are positive real numbers and Z_{ij} are p-dimensional subvarieties of X.

Demailly proved in [Dem82, Théorème 1.10] that, for any smooth projective variety and q as above,

$$\text{HC}^+ \implies \text{HC}.$$

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Furthermore, he showed that HC^+ holds for any smooth projective variety when $q = 1$, see [Dem82, Théorème 1.9] and [Dem12, Chapter 13]. In [Dem12, Theorem 13.40], Demailly showed that, in fact, for any smooth projective variety and q,

$$\mathrm{HC} \iff \mathrm{HC}'.$$

We show that HC^+ is not true in general:

Theorem 0.1. There is a 4-dimensional smooth projective toric variety X and a $(2,2)$-dimensional strongly positive closed current $\mathcal{T}$ on X with the following properties:

(1) The cohomology class of $\mathcal{T}$ satisfies

$$\{\mathcal{T}\} \in H^4(X,\mathbb{Z})/\mathrm{tors} \cap H^{2,2}(X).$$

(2) The current $\mathcal{T}$ is not a weak limit of the form

$$\lim_{i\to\infty} \mathcal{T}_i, \quad \mathcal{T}_i = \sum_j \lambda_{ij}[Z_{ij}],$$

where λ_{ij} are nonnegative real numbers and Z_{ij} are algebraic surfaces in X.

The above current $\mathcal{T}$ generates an extremal ray of the cone of strongly positive closed currents on X: If $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ is any decomposition of $\mathcal{T}$ into strongly positive closed currents, then both $\mathcal{T}_1$ and $\mathcal{T}_2$ are nonnegative multiples of $\mathcal{T}$. This extremality relates to HC^+ by the following application of Milman's converse to the Krein-Milman theorem.

Proposition 0.2. Let X be an algebraic variety and let $\mathcal{T}$ be a (p,p)-dimensional current on X of the form

$$\mathcal{T} = \lim_{i\to\infty} \mathcal{T}_i, \quad \mathcal{T}_i = \sum_j \lambda_{ij}[Z_{ij}],$$

where λ_{ij} are nonnegative real numbers and Z_{ij} are p-dimensional irreducible subvarieties of X. If $\mathcal{T}$ generates an extremal ray of the cone of strongly positive closed currents on X, then there are nonnegative real numbers λ_i and p-dimensional irreducible subvarieties $Z_i \subseteq X$ such that

$$\mathcal{T} = \lim_{i\to\infty} \lambda_i [Z_i].$$

Therefore, if we assume that HC^+ holds for a smooth projective variety X, then every extremal strongly positive closed current with rational cohomology class can be approximated by positive multiples of integration currents along *irreducible* subvarieties of X. Lelong in [Lel73] proved that the integration currents along irreducible analytic subsets are extremal, and asked whether those are the only extremal currents. Demailly in [Dem82] found the first extremal strongly positive closed current on $\mathbb{CP}^2$ with a support of real dimension 3, which, therefore, cannot be an integration current along any analytic set. Later on, Bedford noticed that many extremal currents occur in dynamical systems on several complex variables have fractal sets as their support, and extremal currents of this type were later generated in several works such as [BLS93, Sib99, DS05, Gue05, DS13]. These extremal currents, though, were readily known to be a weak limit of integration currents by the methods of their construction. The first tropical approach to extremal currents was established in the PhD thesis of the first author [Bab14]. He introduced the notion of tropical currents and deduced certain sufficient local conditions which implied extremality.

A *tropical current* is a certain closed current of bidimension (p,p) on the algebraic torus $(\mathbb{C}^*)^n$, which is associated to a tropical variety of dimension p in $\mathbb{R}^n$. A tropical variety is a weighted rational polyhedral complex $\mathcal{C}$ which is *balanced*. The tropical current associated to $\mathcal{C}$, denoted by $\mathcal{T}_{\mathcal{C}}$, has support

$$|\mathcal{T}_{\mathcal{C}}| = \mathrm{Log}^{-1}(\mathcal{C}),$$

where Log is the map defined by

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad \big(z_1, \ldots, z_n\big) \longmapsto \big(-\log|z_1|, \ldots, -\log|z_n|\big).$$

To construct $\mathcal{T}_{\mathcal{C}}$ from a weighted complex $\mathcal{C}$, for each p-dimensional cell σ in $\mathcal{C}$ we consider a current $\mathcal{T}_\sigma$, the average of the integration currents along fibers of a natural fiberation over the real torus $\mathrm{Log}^{-1}(\sigma) \longrightarrow (S^1)^{n-p}$. The current $\mathcal{T}_{\mathcal{C}}$ is then defined by setting

$$\mathcal{T}_{\mathcal{C}} = \sum_\sigma \mathrm{w}_{\mathcal{C}}(\sigma)\mathcal{T}_\sigma,$$

where the sum is over all p-dimensional cells in $\mathcal{C}$ and $\mathrm{w}_{\mathcal{C}}(\sigma)$ is the corresponding weight. We give the following criterion for the closedness of the resulting current $\mathcal{T}_{\mathcal{C}}$, cf. [Bab14, Theorem 3.1.8].

Theorem 0.3. A weighted complex $\mathcal{C}$ is balanced if and only if the current $\mathcal{T}_{\mathcal{C}}$ is closed.

The proof of the above criterion for closedness of $\mathcal{T}_{\mathcal{C}}$, as well as the following criterion for strong extremality of $\mathcal{T}_{\mathcal{C}}$, is based on Fourier analysis on compact tori. A closed current $\mathcal{T}$ with measure coefficients is said to be *strongly extremal* if any closed current $\mathcal{T}'$ with measure coefficients which has the same dimension and support as $\mathcal{T}$ is a constant multiple of $\mathcal{T}$. (Note that if $\mathcal{T}$ is strongly positive and strongly extremal, then $\mathcal{T}$ generates an extremal ray in the cone of strongly positive closed currents.) Similarly, a balanced weighted complex $\mathcal{C}$ is said to be *strongly extremal* if any balanced weighted complex $\mathcal{C}'$ which has the same dimension and support as $\mathcal{C}$ is a constant multiple of $\mathcal{C}$.

Theorem 0.4. A non-degenerate tropical variety $\mathcal{C}$ is strongly extremal if and only if the tropical current $\mathcal{T}_{\mathcal{C}}$ is strongly extremal.

Here a tropical variety in $\mathbb{R}^n$ is said to be *non-degenerate* if its support is contained in no proper subspace of $\mathbb{R}^n$. We note that there is an abundance of strongly extremal tropical varieties. For example, the Bergman fan of any simple matroid is a strongly extremal tropical variety [Huh14, Theorem 38]. There are 376467 nonisomorphic simple matroids on 9 elements [MR08], producing that many strongly extremal strongly positive closed currents on $(\mathbb{C}^*)^8$. By Theorem 0.5 below, all of them have distinct cohomology classes in one fixed toric compactification of

$(\mathbb{C}^*)^8$, the one associated to the permutohedron, see [Huh14] for details. In fact, Demailly's first example of non-analytic extremal strongly positive current in [Dem82] is a tropical current associated to the simplest nontrivial matroid, namely the rank 2 simple matroid on 3 elements.

Let $\overline{\mathcal{T}}_{\mathcal{C}}$ be the trivial extension of the tropical current $\mathcal{T}_{\mathcal{C}}$ to an n-dimensional smooth projective toric variety X whose fan is *compatible* with $\mathcal{C}$. According to Fulton and Sturmfels [FS97], cohomology classes of a complete toric variety bijectively correspond to balanced weighted fans compatible with the fan of the toric variety. We give a complete description of the cohomology class of $\overline{\mathcal{T}}_{\mathcal{C}}$ in X:

Theorem 0.5. If $\mathcal{C}$ is a p-dimensional tropical variety compatible with the fan of X, then

$$\{\overline{\mathcal{T}}_{\mathcal{C}}\} = \mathrm{rec}(\mathcal{C}) \in H^{q,q}(X),$$

where $\mathrm{rec}(\mathcal{C})$ is the *recession* of $\mathcal{C}$. In particular, if all polyhedrons in $\mathcal{C}$ are cones in Σ, then

$$\{\overline{\mathcal{T}}_{\mathcal{C}}\} = \mathcal{C} \in H^{q,q}(X).$$

This shows that every cohomology class of a smooth projective toric variety has a canonical representative in the space of closed currents. The current $\mathcal{T}$ in Theorem 0.1 is a current of the form $\overline{\mathcal{T}}_{\mathcal{C}}$, and Theorem 0.5 plays an important role in justifying the claimed properties of $\mathcal{T}$.

The proof of Theorem 0.1 is completed by analyzing a certain Laplacian matrix associated to a 2-dimensional tropical variety $\mathcal{C}$. According to Theorem 0.5, if $\mathcal{C}$ is compatible with the fan of an n-dimensional smooth projective toric variety X, we may view the cohomology class of $\overline{\mathcal{T}}_{\mathcal{C}}$ as a geometric graph $G = G(\mathcal{C}) \subseteq \mathbb{R}^n \setminus \{0\}$ with edge weights w_{ij} satisfying the *balancing condition*: At each of its vertex u_i there is a real number d_i such that

$$d_i u_i = \sum_{u_i \sim u_j} w_{ij} u_j,$$

where the sum is over all neighbors of u_i in G. We define the *tropical Laplacian* of $\mathcal{C}$ to be the real symmetric matrix L_G with entries

$$(L_G)_{ij} := \begin{cases} d_i & \text{if } u_i = u_j, \\ -w_{ij} & \text{if } u_i \sim u_j, \\ 0 & \text{if otherwise,} \end{cases}$$

where the diagonal entries d_i are the real numbers satisfying

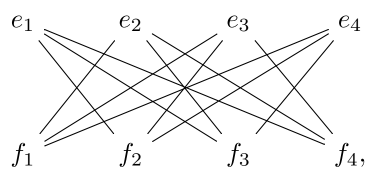
$$d_i u_i = \sum_{u_i \sim u_j} w_{ij} u_j.$$

When G is the graph of a polytope with weights given by the Hessian of the volume of the dual polytope, the matrix L_G has been considered in various contexts related to rigidity and polyhedral combinatorics [Con82, Fil92, Lov01, Izm10]. In this case, L_G is known to have exactly one negative eigenvalue, by the Alexandrov-Fenchel inequality. See, for example, [Fil92, Proposition 4] and [Izm10, Theorem A.10]. Using the Hodge index theorem and the continuity of the cohomology class assignment, we show that L_G has at most one negative eigenvalue if $\overline{\mathcal{T}}_{\mathcal{C}}$ is a weak limit of integration currents along irreducible surfaces in X.

The remaining step is to the construct of a strongly extremal tropical surface $\mathcal{C}$ whose tropical Laplacian has more than one negative eigenvalue. For this we introduce two 'flip' operations on weighted fans, $F \longmapsto F_{ij}^{+}$ and $F \longmapsto F_{ij}^{-}$, and repeatedly apply them to the graph of a 'twisted cube' in $\mathbb{R}^4$ to arrive at $\mathcal{C}$ with the desired properties. The eight vertices e_1, e_2, e_3, e_4, and f_1, f_2, f_3, f_4 of the twisted cube are the rows of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix},$$

and its abstract graph is



After the flip operation needed to eliminate edges with negative weights, we get a geometric graph whose tropical Laplacian is the symmetric matrix

$$L = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 3 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

The matrix has 7 positive eigenvalues and 3 negative eigenvalues. By the above Theorems 0.3, 0.4, and 0.5, the corresponding tropical current $\overline{\mathcal{T}}_{\mathcal{C}}$ is strongly extremal strongly positive closed current which is not a weak limit of positive linear combinations of integration currents along subvarieties.

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