NEGATIVE CORRELATION AND HODGE-RIEMANN RELATIONS

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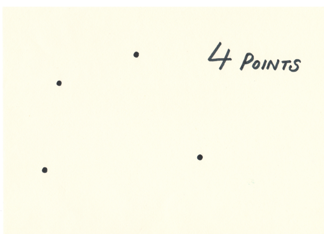
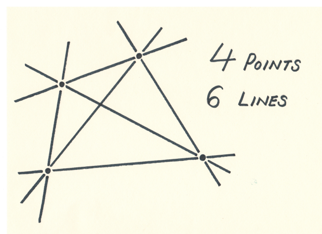
1. Hard Lefschetz

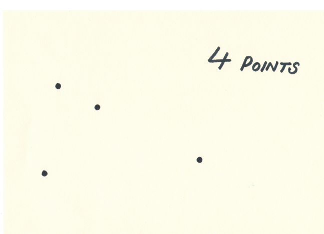
1.1. Here are 4 points in projective plane:

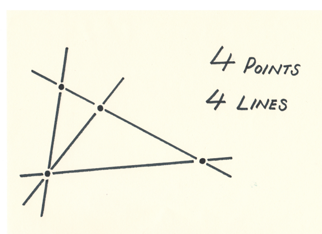




The 4 points determine 6 lines.

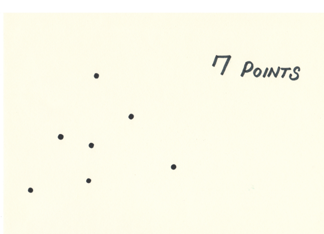
Let's move one of the points into special position:

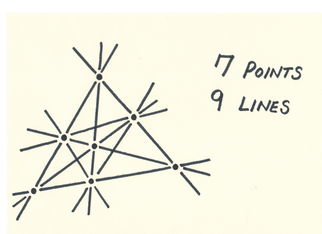




Now the 4 points determine 4 lines.

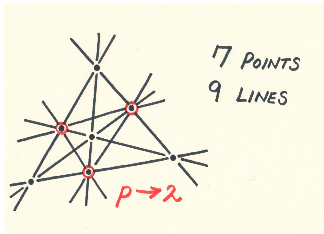
Think of all seven points with zero-one coordinates:

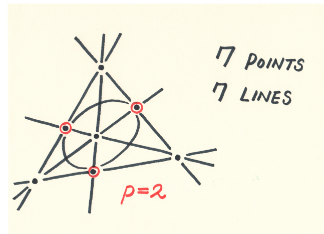




The 7 points determine 9 lines.

What happens if we move "p" in the base $\mathrm{Spec}(\mathbb{Z})$ toward the prime number 2?



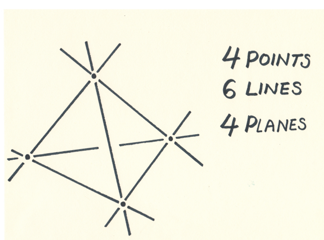


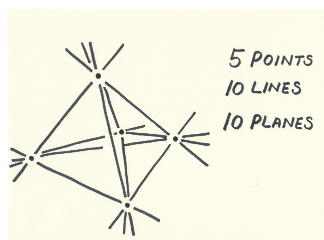
The 7 points determine 7 lines.

Here is a theorem of de Bruijn and Erdős from 1948 [dBE48]:

Every set of points E in a projective plane determines at least $|E|$ lines, unless E is contained in a line.

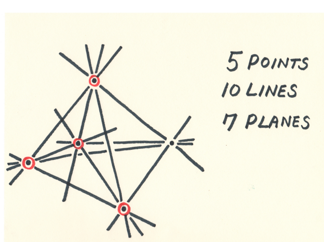
1.2. Here are 4 points in space defining 6 lines and 4 planes:

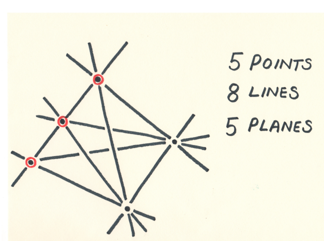




. . . and 5 points defining 10 lines and 10 planes.

Here are 5 points defining 10 lines and 7 planes:





. . . and 5 points defining 8 lines and 5 planes.

Here is a theorem of Motzkin from 1951:

> *Every set of points E in a projective space determines at least $|E|$ hyperplanes, unless E is contained in a hyperplane.*

Motzkin worked over the real numbers and used properties of real numbers [Mot51]. The assumption was removed by Basterfield and Kelly in 1968 [BK68].

1.3. Let E be a spanning subset of a d-dimensional vector space V, and let $\mathscr{L}$ be the poset of subspaces of V spanned by subsets of E. Write $\mathscr{L}_p$ for the set of p-dimensional spaces in $\mathscr{L}$.

Example. If E is the set of 4 general vectors in $\mathbb{R}^3$, then

$$|\mathscr{L}_0| = 1,$$
$$|\mathscr{L}_1| = 4,$$
$$|\mathscr{L}_2| = 6,$$
$$|\mathscr{L}_3| = 1.$$

Here is a conjecture of Dowling and Wilson from 1974 [DW74, DW75]: *For every $p \leqslant \frac{d}{2}$, we have*

$$|\mathscr{L}_p| \leqslant |\mathscr{L}_{d-p}|.$$

The case of de Bruijn and Erdős is $d = 3$ and $p = 1$.

Theorem (with Botong Wang [HW])**.** For every $p \leqslant \frac{d}{2}$, there is an injective map

$$\mathscr{L}_p \longrightarrow \mathscr{L}_{d-p}.$$

that respects incidence relations between subspaces.

No combinatorial proof is known; Dowling-Wilson conjecture for non-realizable matroids remain open. The current proof in the realizable case uses the decomposition theorem and hard Lefschetz theorem for ℓ-adic perverse sheaves.

2. Hodge-Riemann

A *matroid* M on a finite set E is a nonempty collection $\mathscr{B}$ of subsets of E that satisfies the *basis exchange axiom*:

> *"For any $B_1, B_2 \in \mathscr{B}$ and any $x_1 \in B_1 \backslash B_2$, there is $x_2 \in B_2 \backslash B_1$ such that $B_1 \backslash x_1 \cup x_2 \in \mathscr{B}$."*

Three types of matroids:

(1) $G =$ connected graph, $E =$ edges of G, $\mathscr{B} =$ spanning trees of G.

(2) $V =$ vector space, $E =$ spanning subset of V, $\mathscr{B} =$ bases of V in E.

(3) Matroids not of type (1), (2).

For a matroid M on $\{1, 2, \ldots, n\}$, define an $n \times n$ matrix $B(M)$ by

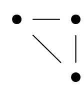
$$B(M)_{ij} = \begin{cases} 0 & \text{if } i = j, \\ b_{ij} & \text{if } i \neq j, \end{cases}$$

where b_{ij} is the number of elements of $\mathscr{B}$ containing i and j. To avoid trivialities, suppose $B(M) \neq 0$.

Theorem (with Botong Wang)**.** The matrix $B(M)$ has exactly one positive eigenvalue.

The proof is based on the Hodge-Riemann relation for matroids proved in [AHK].

Example. Let M be the matroid of the complete graph on three vertices:

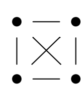


Then the matrix associated to M is

$$B(M) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $B(M)$ are $2, -1, -1$.

Example. Let M be the matroid of the complete graph on four vertices:



Then the matrix associated to M is

$$B(M) = \begin{pmatrix} 0 & 3 & 3 & 3 & 3 & 4 \\ 3 & 0 & 3 & 3 & 4 & 3 \\ 3 & 3 & 0 & 4 & 3 & 3 \\ 3 & 3 & 4 & 0 & 3 & 3 \\ 3 & 4 & 3 & 3 & 0 & 3 \\ 4 & 3 & 3 & 3 & 3 & 0 \end{pmatrix}.$$

The eigenvalues of $B(M)$ are $16, -2, -2, -4, -4, -4$.

3. NEGATIVE CORRELATION

Let G be a finite connected graph, and let T be a random spanning tree of G. Then, for any distinct edges i and j,

$$\Pr(i \in T) \geqslant \Pr(i \in T \mid j \in T).$$

In other words, for any distinct edges i and j,

$$\frac{b_i}{b} \geqslant \frac{b_{ij}}{b_j}.$$

The same must be true for any matroid, right?

Example. In 1974, Seymour and Welsh found M over characteristic 2 which has

$$\frac{b\, b_{ij}}{b_i b_j} \simeq 1.02\ldots \quad \text{for some } i \text{ and } j.$$

Let's view the symmetric matrix $B(M)$ as a bilinear form

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Restrict the form to the 3-dimensional space spanned by e_i, e_j, and $e_1 + \cdots + e_n$ to get a 3×3 matrix $B_{ij}(M)$. Then the Cauchy interlacing implies:

$B_{ij}(M)$ is either zero or has exactly one positive eigenvalue.

Therefore, for any M and any distinct i and j, we have

$$\det B_{ij}(M) \geqslant 0.$$

We may rewrite the condition on the determinant by

$$\frac{\text{rk}(M)}{\text{rk}(M)-1}\,\frac{b\, b_{ij}}{b_i b_j} \leqslant 2.$$

Therefore, for any M and any distinct i and j,

$$\frac{b\, b_{ij}}{b_i b_j} < 2.$$

Question. How large can the ratio $\frac{b\, b_{ij}}{b_i b_j}$ be over a given field?

This may be an interesting invariant of a field. The current record is $1.142\ldots$ for $\mathbb{Z}/2\mathbb{Z}$ by Benjamin Schröter at Berlin.

References

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[Mot51] Theodore Motzkin, *The lines and planes connecting the points of a finite set*. Trans. Amer. Math. Soc. **70** (1951), 451–464. 3