Tropical homology and Betti numbers of real algebraic varieties

Ilia Itenberg

Abstract

Under some assumptions, a tropical variety can be approximated by a one-parametric family of complex varieties, which provides an important link between complex and tropical geometries. The purpose of this talk is to discuss tropical homology together with its relations to Hodge decompositions (respectively, homology) in complex (respectively, real) world.

1 Combinatorial patchworking

One of the motivations for the consideration of tropical homology comes from the *combinatorial patchworking.* This construction is one of the sources of tropical geometry and is a particular case of the *Viro method* of construction of real algebraic varieties with prescribed topology. We start with a brief description of the construction of combinatorial patchworking restricting ourselves to the case of hypersurfaces in real projective spaces.

Let n and d be positive integer numbers (they are, respectively, the dimension of the ambient projective space and the degree of the hypersurface under construction), and let $T^n(d)$ be the simplex in $\mathbb{R}^n$ with vertices

$$(0,0,\dots,0),\ (0,0,\dots,0,d),\ (0,\dots,0,d,0),\ \dots,(d,0,\dots,0).$$

We shorten the notation of $T^n(d)$ to T, when n and d are unambiguous and call $T^n(d)$ the *standard n-simplex of size d.* Take a triangulation τ of T with vertices having integer coordinates. Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates $(i_1,\dots,i_n)$ is denoted by $\alpha_{i_1,\dots,i_n}$.

Denote by T_* the union of all symmetric copies of T under reflections and compositions of reflections with respect to coordinate hyperplanes. Extend the triangulation τ to a symmetric triangulation τ_* of T_*, and the distribution of signs $\alpha_{i_1,\dots,i_n}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate hyperplane we preserve the sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If an n-simplex of the triangulation of T_* has vertices of different signs, select a piece of hyperplane being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by H the union of the selected pieces. It is a piecewise-linear hypersurface contained in T_*. It is not a simplicial subcomplex of T_*, but it can be deformed by an isotopy preserving τ_* to a subcomplex K of the first barycentric subdivision τ'_* of τ_*. Each n-simplex of τ'_* has a unique vertex belonging to τ_*. Denote by τ_*^+ the union of all the n-simplices of τ'_* containing positive vertices of τ_* and by τ_*^- the union of all the rest n-simplices. The subcomplex K is the intersection of τ_*^+ and τ_*^-. A point of H contained in a simplex σ of τ_* belongs to a unique segment connecting the face of σ with positive vertices and the face with negative ones. This segment meets K also in a unique point and the deformation of H to K can be done along those segments.

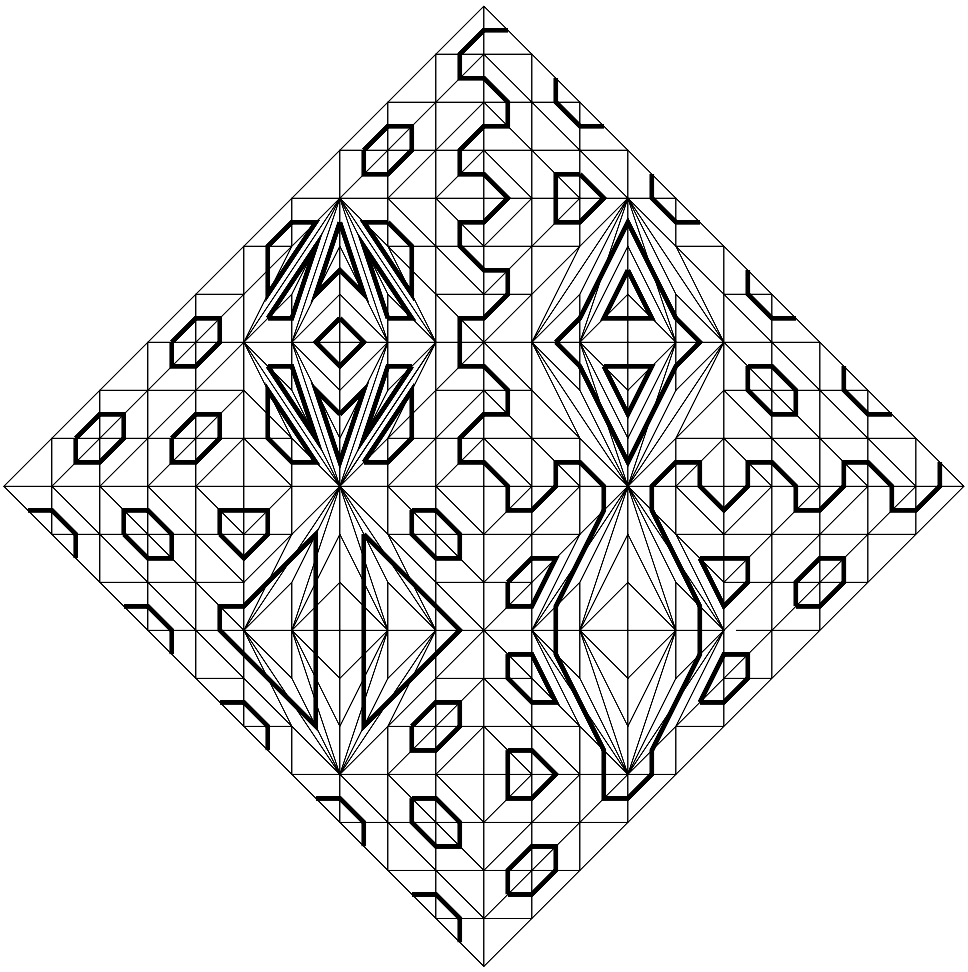

Figure 1: Example of combinatorial patchworking

Identify by the symmetry with respect to the origin the faces of T_*. The quotient space $\widetilde{T}$ is homeomorphic to the real projective space $\mathbb{R}P^n$. Denote by $\widetilde{H}$ the image of H in $\widetilde{T}$.

A triangulation τ of T is said to be *convex* if there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the n-simplices of τ. Sometimes, such triangulations are also called *coherent* (see [GKZ94]) or *regular* (see [Zie94]).

Theorem 1.1 (Viro theorem) (see [V83, V84]) *If τ is convex, there exists a non-singular hypersurface X of degree d in $\mathbb{R}P^n$ and a homeomorphism $\mathbb{R}P^n \to \widetilde{T}$ mapping the set of real points $\mathbb{R}X$ of X onto $\widetilde{H}$.*

The statement of the above theorem can be naturally generalized replacing the simplex T by an arbitrary polytope Δ with integer vertices in $\mathbb{R}^n$ and replacing the projective space by the toric variety associated with Δ.

A hypersurface X with the properties described in Theorem 1.1 can be presented by polynomial

$$\sum_{(i_1,\ldots,i_n)\in V} \alpha_{i_1,\ldots i_n} t^{-\nu(i_1,\ldots i_n)} x_0^{d-i_1-\ldots i_n} x_1^{i_1} \ldots x_n^{i_n},$$

where V is the set of vertices of τ, and t is a positive and sufficiently big real number. For function ν we can take any convex piecewise-linear function certifying that the triangulation τ is convex.

The piecewise-linear hypersurface $\widetilde{H}$ which appears in the statement of Theorem 1.1 is directly related to the tropical hypersurface $\mathcal{H}$ defined in $\mathbb{R}^n$ by the tropical polynomial

$$\max_{(i_1,\ldots,i_n)\in V} \{-\nu(i_1,\ldots i_n) + i_1 a_1 + \ldots + i_n a_n\}.$$

The triangulation τ is the dual subdivision of $\mathcal{H}$. For each quadrant $Q \subset \mathbb{R}^n$, the intersection of $\widetilde{H}$ with the interior of Q can be identified with an appropriate subset of $\mathcal{H}$ (details can be found, for example, in [BIMS15]).

If the triangulation τ is *primitive* (that is, all n-simplices of τ are of volume $1/n!$; this is the minimal possible volume for an n-simplex with vertices having integer coordinates), then the tropical hypersurface $\mathcal{H}$ is said to be *non-singular*.

A hypersurface X defined in $\mathbb{R}P^n$ by polynomial

$$\sum_{(i_1,\ldots,i_n)\in V} \alpha_{i_1,\ldots i_n} t^{-\nu(i_1,\ldots i_n)} x_0^{m-i_1-\ldots i_n} x_1^{i_1} \ldots x_n^{i_n},$$

where V is the set of vertices of τ, and t is a positive and sufficiently big real number, is called a *real subtropical hypersurface* (or *T-hypersurface*) in $\mathbb{R}P^n$. If τ is primitive, we speak about primitive real subtropical hypersurface in $\mathbb{R}P^n$.

2 Topology of real algebraic hypersurfaces in $\mathbb{R}P^n$

In 1876 A. Harnack published a paper [Har76] where he found an exact upper bound for the number of connected components of the set of real points of a curve of a given degree in $\mathbb{R}P^2$. Harnack proved that the number of components of the set of real points of a real plane projective curve of degree d is at most $\frac{(d-1)(d-2)}{2} + 1$. On the other hand, for each natural number d he constructed a non-singular real projective curve of degree d with $\frac{(d-1)(d-2)}{2} + 1$ components of the set of real points, which shows that his estimate cannot be improved without introducing new ingredients. From Harnack's results it is easy to deduce a *topological classification* of non-singular curves of degree d in $\mathbb{R}P^2$ (that is, a classification of the real point sets of these curves up to homeomorphism): the set of real points of such a curve is a collection of circles embedded in $\mathbb{R}P^2$ and the number ℓ of these circles can be any integer between 0 (respectively, 1) and $\frac{(d-1)(d-2)}{2} + 1$ if d is even (respectively, odd).

A much more difficult question (included by D. Hilbert in the 16-th problem of his list) concerns *isotopy classification* of non-singular curves of degree d in $\mathbb{R}P^2$ (that is, a classification of the real point sets of these curves up to isotopy in $\mathbb{R}P^2$). Since any self-homeomorphism of $\mathbb{R}P^2$ is isotopic to identity, the isotopy classification of non-singular curves of degree d in $\mathbb{R}P^2$ is a classification of topological pairs $(\mathbb{R}P^2, \mathbb{R}X)$ up to homeomorphism, where $\mathbb{R}X$ is the set of real points of a non-singular curve of degree d in $\mathbb{R}P^2$. Such a classification is known only for $d \leq 7$.

Harnack [Har76] asked whether the inequality he proved in the case of curves has an analog for surfaces in the three-dimensional projective space. This question is known as *Harnack problem*. Understood literally, *i.e.*, as a question about the number of connected components of the set of real points, it has appeared to be a difficult problem. The maximal number of components is found only for degree ≤ 4 (for degree 5 this maximal number is known to be either 23, or 24, or 25). However, the Harnack inequality can be generalized in other way.

Theorem 2.1 (Smith-Thom inequality) *If X is an algebraic variety, then*

$$\sum_{i \geq 0} \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(\mathbb{R}X;\, \mathbb{Z}/2\mathbb{Z}) \leq \sum_{i \geq 0} \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(\mathbb{C}X;\, \mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{R}X$ and $\mathbb{C}X$ are the sets of real and complex points of X, respectively.

A real algebraic variety for which the left and right hand sides of the Smith-Thom inequality are equal is called an *M-variety* (or a *maximal variety*).

Put $b_i(Y) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_i(Y; \mathbb{Z}/2\mathbb{Z})$ for any topological space Y and any non-negative integer i. If X is a non-singular hypersurface of degree d in $\mathbb{R}P^n$, the Smith-Thom inequality provides an upper bound for every Betti number $b_i(\mathbb{R}X)$ of $\mathbb{R}X$. However, if $n \geq 3$, these upper bounds are, in general, far from being sharp. The question about sharp upper bounds for the individual Betti numbers of real point sets of hypersurfaces of given degree in $\mathbb{R}P^n$ is one of the main problems in topology of real algebraic varieties.

It is interesting to consider the problem formulated above near the tropical limit. Here are two facts concerning the Betti numbers of primitive real subtropical hypersurfaces in $\mathbb{R}P^n$.

Theorem 2.2 (see [IV07]) *Let $n \geq 2$ be an integer, and let $(X_d)_{d \geq 1}$ be a sequence, where X_d is a primitive real subtropical hypersurface of degree d in $\mathbb{R}P^n$. Then, for any integer $0 \leq i \leq n-1$, one has*

$$b_i(\mathbb{R}X_d) \leq h^{i,n-1-i}(\mathbb{C}X_d) + O(d^{n-1}),$$

where $h^{i,n-1-i}$ are Hodge numbers.

Remark 2.3 *Let $n \geq 2$ and $d \geq 1$ be integers. For any integer $i = 0, \ldots, n-1$, put*

$$H_i(n,d) = \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \binom{d(i+1) - (d-1)j - 1}{n}.$$

If X is a non-singular hypersurface of degree d in $\mathbb{C}P^n$, then for any integer $0 \leq i \leq n-1$, one has $H_i(n,d) = h^{i,n-1-i}(\mathbb{C}X) - 1$, if $n-1 = 2i$, and $H_i(n,d) = h^{i,n-1-i}(\mathbb{C}X)$, otherwise (see [DKh86]). In particular, for any integer $i = 0, \ldots, n-1$, there exists a non-zero constant c_i such that

$$h^{i,n-1-i}(\mathbb{C}X) = c_i d^n + O(d^{n-1}).$$

Theorem 2.4 (see [B10]) *Let $n \geq 2$ and $d \geq 1$ be integers, and let X be a primitive real subtropical hypersurface of degree d in $\mathbb{R}P^n$. Then, the Euler characteristic $\chi(\mathbb{R}X)$ of $\mathbb{R}X$ is equal to the signature $\sigma(\mathbb{C}X)$ of $\mathbb{C}X$. (If n is even, we set the signature of $\mathbb{C}X$ to be 0.)*

In the statement of Theorem 2.4, the projective space can be replaced by any non-singular projective toric variety (see [B10]). Very recently, this statement was reproved by Ch. Arnal [A17] using the tropical homology. The above statement can be generalized to the case of primitive real subtropical complete intersections (see [BB07, A17]).

The tropical homology seems to be an appropriate tool to study Betti numbers of real algebraic varieties near the tropical limit, and in particular, to attack the following conjecture.

Conjecture 2.5 *Let $n \geq 2$ and $d \geq 1$ be integers, and let X be a primitive real subtropical hypersurface of degree d in $\mathbb{R}P^n$. Then, for any integer $i = 0, \ldots, n-1$, one has*

$$b_i(\mathbb{R}X) \leq \begin{cases} h^{i,n-1-i}(\mathbb{C}X), & \textit{if} \quad i = (n-1)/2, \\ h^{i,n-1-i}(\mathbb{C}X) + 1, & \textit{otherwise.} \end{cases}$$

In the case $n = 2$, the statement of this conjecture is equivalent to Harnack's inequality. In the case $n = 3$, the statement of the conjecture follows from the Smith-Thom inequality and Theorem 2.4. In the case of arbitrary non-singular algebraic surfaces in $\mathbb{R}P^3$, the inequalities $b_0(\mathbb{R}X) \leq h^{0,2}(\mathbb{C}X) + 1$ and $b_1(\mathbb{R}X) \leq h^{1,1}(\mathbb{C}X)$ are wrong in general. The first inequality is wrong already for surfaces of degree 4, since the real point set of such a surface can have up to 10 connected components. The second inequality is the so-called *Viro conjecture*. It is directly related to the Ragsdale conjecture [Ra06]; counter-examples can be found in [I93, I97]. It is interesting to notice that the counter-examples in [I97] are constructed via the combinatorial patchworking using a non-primitive triangulation.

3 Tropical homology

Tropical varieties are certain finite-dimensional polyhedral complexes enhanced with the *tropical structure*. This is a geometric structure that can be thought of as a version of an affine structure for polyhedral complexes. For example, the *tropical projective space*

$$\mathbb{T}P^n = (\mathbb{T}^{n+1} \setminus \{(-\infty, \ldots, -\infty)\})/ \sim,$$

where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ is the *tropical semi-field* and $(x_0, \ldots, x_n) \sim (x_0 + \lambda, \ldots, x_n + \lambda)$ for any $\lambda \in \mathbb{R}$, is a smooth projective tropical variety homeomorphic to an n-simplex. The restriction of the tropical structure to the relative interior of every k-dimensional face σ of $\mathbb{T}P^n$ turns σ into $\mathbb{R}^k$ (with the tautological affine structure of $\mathbb{R}^k = \mathbb{Z}^k \otimes \mathbb{R}$). A projective tropical m-variety $\mathcal{X}$ is a certain m-dimensional polyhedral complex in $\mathbb{T}P^n$. For example, the tropical hypersurface $\mathcal{H}$ considered in Section 1 is, in fact, a hypersurface in $\mathbb{T}P^n$ (in Section 1 we spoke about the intersection of this hypersurface with the n-dimensional stratum $\mathbb{R}^n$ of $\mathbb{T}P^n$).

A tropical structure on a tropical variety $\mathcal{X}$ can be used to define a natural coefficient system ${}^{\mathbb{Z}}\mathcal{F}_p$ (see the details below). This system is not locally constant everywhere, but it is constant on the relative interiors of faces of $\mathcal{X}$. Furthermore, it is a constructible cosheaf of abelian groups. The tropical (p, q)-homology group $H_{p,q}(\mathcal{X})$ is the q-dimensional homology group of $\mathcal{X}$ with coefficients in $\mathcal{F}_p = {}^{\mathbb{Z}}\mathcal{F}_p \otimes \mathbb{Q}$.

An important example of projective tropical varieties is provided by the *tropical limit* of an algebraic family $Z_w \subset \mathbb{C}P^n$, $w \in \mathbb{C}$, $t = |w| \to +\infty$, of complex projective m-dimensional varieties. It may be shown (cf. e.g. the fundamental theorem of tropical geometry in [MS15]) that the sets $\mathrm{Log}_t(Z_w) \subset \mathbb{T}P^n$, where the map $\mathrm{Log}_t : \mathbb{C}P^n \to \mathbb{T}P^n$ is defined by $\mathrm{Log}_t(z_0 : \ldots : z_n) \mapsto (\log_t |z_0| : \ldots : \log_t |z_n|)$, converge to an m-dimensional balanced weighted polyhedral complex $\mathcal{X}$ in $\mathbb{T}P^n$. If $\mathcal{X}$ is a smooth tropical variety (for the definition, see, for example, [BIMS15]), then for a generic w the complex variety Z_w is smooth. Notice that the above example generalizes the tropical limits of combinatorial patchworking families (if a piecewise-linear convex function certifying the convexity of the triangulation is chosen in an appropriate way).

The following result establishes, under certain assumptions, the equality between $\dim H_{p,q}(\mathcal{X})$ and the Hodge numbers $h^{p,q}(Z_w)$.

Theorem 3.1 (see [IKMZ16]) *Let $\mathcal{Z} \subset \mathbb{C}P^n \times \mathcal{D}^*$ be a complex analytic one-parameter family of projective varieties over the punctured disc $\mathcal{D}^*$. Assume that $\mathcal{Z}$ admits a tropical limit $\mathcal{X} \subset \mathbb{T}P^n$, which is a smooth projective $\mathbb{Q}$-tropical variety (the prefix $\mathbb{Q}$- means that all the inequalities defining faces of $\mathcal{X}$ have rational coefficients; for precise definitions see [IKMZ16]). Then, the dual spaces $\mathrm{Hom}(H_q(\mathcal{X}; \mathcal{F}_p), \mathbb{Q})$ to the tropical homology groups $H_q(\mathcal{X}; \mathcal{F}_p)$ are naturally isomorphic to the associated graded pieces W_{2p}/W_{2p-1} of the weight filtration in the limiting mixed Hodge structure on $H^{p+q}(\mathcal{Z}_\infty, \mathbb{Q})$, where $\mathcal{Z}_\infty$ is the canonical fiber of the family $\mathcal{Z}$.*

Under the assumptions of Theorem 3.1, the limiting mixed Hodge structure is of Hodge-Tate type. That is, only even associated graded pieces $\mathrm{Gr}^W_{2p} H^k(\mathcal{Z}_\infty; \mathbb{Q}) = W_{2p}/W_{2p-1}$ are non-trivial and they have Hodge (p,p)-type. Hence, the Hodge numbers $h^{p,q}(Z_w)$ agree with the dimensions of the spaces $\mathrm{Gr}^W_{2p} H^{p+q}(\mathcal{Z}_\infty; \mathbb{Q})$.

Corollary 3.2 *The Hodge numbers $h^{p,q}(Z_w)$ of a general fiber equal the dimensions of the tropical homology groups $H_q(\mathcal{X}; \mathcal{F}_p)$.*

The proof of Theorem 3.1 consists of providing a quasi-isomorphism between the tropical cellular complexes and the dual row complexes of the E^1-term of the Steenbrink-Illusie spectral sequence (see, for example, [PS08]) for the limiting mixed Hodge structure. Various results related to Theorem 3.1 appeared already in the literature (see, for example, [KS16] which contains a relevant statement in the case of hypersurfaces).

We now give detailed definitions of the coefficient system ${}^{\mathbb{Z}}\mathcal{F}_p$ and the tropical homology groups. Let $\Sigma = \bigcup \sigma \subset \mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R}$ be a rational polyhedral fan. For each cone $\sigma \subset \Sigma$, we denote by $< \sigma >_{\mathbb{Z}}$ the integral lattice in the vector subspace linearly spanned by σ.

Definition 3.3 *For any integer $p \geq 0$, the group ${}^{\mathbb{Z}}\mathcal{F}_p(\Sigma)$ is the subgroup of $\wedge^p \mathbb{Z}^n$ generated by the elements $v_1 \wedge \cdots \wedge v_p$, where all $v_1, \ldots, v_p \in < \sigma >_{\mathbb{Z}}$ for some cone $\sigma \in \Sigma$. It is important that all p vectors v_i come from the* same *cone. The dual group ${}^{\mathbb{Z}}\mathcal{F}^p(\Sigma) = \mathrm{Hom}({}^{\mathbb{Z}}\mathcal{F}_p(\Sigma), \mathbb{Z})$ is the quotient of $\wedge^p(\mathbb{Z}^n)^*$ by $({}^{\mathbb{Z}}\mathcal{F}_p(\Sigma))^{\perp}$.*

It is not hard to see that the groups ${}^{\mathbb{Z}}\mathcal{F}^p(\Sigma)$ form a graded algebra ${}^{\mathbb{Z}}\mathcal{F}^{\bullet}(\Sigma)$ over $\mathbb{Z}$ under the wedge product in $\wedge^k(\mathbb{Z}^N)^*$.

Let $\mathcal{X} \subset \mathbb{T}P^n$ be a smooth projective $\mathbb{Q}$-tropical variety. The polyhedral decomposition of $\mathcal{X}$ into faces gives it a natural cell structure. To simplify the presentation, we assume that each face of $\mathcal{X}$ is entirely contained in at least one affine chart of $\mathbb{T}P^n$. We say that a point $x \in \mathbb{T}P^n$ is of *sedentarity* $I \subset \{0, 1, \ldots, n\}$ if I is the set of indices of those coordinates of x that are equal to $-\infty$. The sedentarity of a point $x \in \mathbb{T}P^n$ is sometimes denoted by $I(x)$. For any subset $I \subsetneq \{0, 1, \dots, n\}$, denote by $\mathbb{T}_I^0 \subset \mathbb{T}P^n$ the subset formed by the points of sedentarity I. The sedentarity of a face of $\mathcal{X}$ is the sedentarity of the points of the relative interior of this face.

Let $x \in \mathcal{X}$ be a point in the relative interior of a face Δ_x of sedentarity I in $\mathcal{X}$. We define $\Sigma(x)$, the *fan at* x, to be the cone in $\mathbb{T}_I^\circ \cong \mathbb{R}^{n-|I|}$ (where $|I|$ is the number of elements in I) consisting of vectors $u \in \mathbb{T}_I^\circ$ such that $x + \epsilon u \in \mathcal{X} \cap \mathbb{T}_I^\circ$ for a sufficiently small $\epsilon > 0$ (depending on u).

Definition 3.4 *For any integer $p \geq 0$, we define the coefficient groups $\mathcal{F}_p(x)$ and $\mathcal{F}^p(x)$ to be ${}^{\mathbb{Z}}\mathcal{F}_p(\Sigma(x)) \otimes \mathbb{Q}$ and ${}^{\mathbb{Z}}\mathcal{F}^p(\Sigma(x)) \otimes \mathbb{Q}$, respectively.*

Note that the groups $\mathcal{F}_p(x)$ and $\mathcal{F}_p(y)$ are canonically identified by translation if x and y belong to the relative interior of the same face Δ of X. Thus, we can use the notation $\mathcal{F}_p(\Delta)$.

Let x and y be two points of $\mathcal{X}$ such that the face Δ_y whose relative interior contains y is a face of the face Δ_x whose relative interior contains x. Then, there are natural homomorphisms

$$\iota : \mathcal{F}_p(x) \to \mathcal{F}_p(y). \tag{1}$$

To define the maps (1) we take an affine chart $U^{(i)} \ni y$. If $I(y) = I(x)$, then any face adjacent to x is contained in some face adjacent to y, and the inclusion induces the required map. If $I(y) \neq I(x)$ (note that we must have $I(y) \supset I(x)$), then the required map is given by the projection along the directions in $U^{(i)}$ that are indexed by $I(y) \setminus I(x)$.

For a pair of adjacent faces Δ and Δ' of $\mathcal{X}$, where Δ is a face of Δ', the map (1) and its dual can be rewritten as

$$\iota : \mathcal{F}_p(\Delta') \to \mathcal{F}_p(\Delta), \qquad \iota^* : \mathcal{F}^p(\Delta) \to \mathcal{F}^p(\Delta'). \tag{2}$$

This allows us to define a complex $C_\bullet(\mathcal{X}; \mathcal{F}_p)$, where

$$C_q(\mathcal{X}; \mathcal{F}_p) = \oplus \mathcal{F}_p(\Delta).$$

Here, the direct sum is taken over all q-dimensional faces of X. We can write a chain in $C_q(\mathcal{X}; \mathcal{F}_p)$ as $\sum \beta_\Delta \Delta$. The boundary map

$$\partial : C_q(\mathcal{X}; \mathcal{F}_p) \to C_{q-1}(\mathcal{X}; \mathcal{F}_p)$$

is the usual cellular boundary combined with the maps ι in (2) for any pair of faces Δ and Δ' of $\mathcal{X}$ such that Δ is a face of codimension 1 of Δ'. The groups

$$H_q(\mathcal{X}; \mathcal{F}_p) = H_q(C_\bullet(\mathcal{X}; \mathcal{F}_p), \partial)$$

are called the (*cellular*) *tropical* (p, q)*-homology* groups.

We can consider the dual cochain complex $C^\bullet(\mathcal{X}; \mathcal{F}^p)$ of linear functionals on faces Δ of $\mathcal{X}$ with values in $\mathcal{F}^p(\Delta)$ and define the differential δ as the usual coboundary combined with the maps ι^* in (2). This defines the (*cellular*) *tropical* (p, q)*-cohomology* groups

$$H^q(\mathcal{X}; \mathcal{F}^p) = H^q(C^\bullet(\mathcal{X}; \mathcal{F}^p), \delta).$$

Some important properties of tropical homology and cohomology, as well as connections with cohomology of superforms on polyhedral complexes and differential forms on Berkovich spaces (see [CLD12]), can be found in [AB14] and [JSS15].

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Institut de Mathématiques de Jussieu - Paris Rive Gauche
Université Pierre et Marie Curie
4 place Jussieu, 75252 Paris Cedex 5, France
and Département de mathématiques et applications
Ecole Normale Supérieure
45 rue d'Ulm, 75230 Paris Cedex 5, France
E-mail address: `ilia.itenberg@imj-prg.fr`