TROPICAL BRILL-NOETHER THEORY AND APPLICATIONS II

DAVID JENSEN

These notes are from the second of two talks related to ongoing joint work with Sam Payne. The reader will likely find it helpful to consult the notes from the first talk prior to reading these.

1. Motivation

Many interesting questions one can ask about an algebraic curve X concern the ranks of maps between linear series on X. For example, a well-known theorem of Gieseker states the following.

Gieseker-Petri Theorem. [Gie82] Let X be a general curve of genus g. Then $\mathcal{G}^r_d(X)$ is smooth.

By computing the Zariski tangent space to the variety $\mathcal{G}^r_d(X)$, one sees that the Gieseker-Petri Theorem is equivalent to the statement that the natural map

$$\mu : \mathcal{L}(D) \otimes \mathcal{L}(K_X - D) \to \mathcal{L}(K_X)$$

is injective for all divisors D on a general curve X.

One of the major open problems in Brill-Noether theory is similarly concerned with the ranks of maps between linear series.

Maximal Rank Conjecture. Let $V \subset \mathcal{L}(D)$ be a general linear series of rank r and degree d on a general curve X of genus g. Then the multiplication maps

$$\mu_m : Sym^m \mathcal{L}(D) \to \mathcal{L}(mD)$$

have maximal rank for all m. That is, they are either injective or surjective.

In this talk, we will demonstrate how tropical methods can be used to approach the types of problems described above.

2. Tropical Independence

Our approach to showing that a linear map has rank at least k is in some sense straightforward; we simply choose k elements of the image, and then show that they are linearly independent. Using tropical methods, we provide a combinatorial obstruction to linear dependence of rational functions on an algebraic curve in terms of their specializations to the skeleton. More precisely, given a nonzero function $f \in K(X)^*$, one can define a piecewise linear function $\mathrm{trop}(f)$ on $\Gamma = Sk(X^{\mathrm{an}})$ by $y \mapsto \mathrm{val}_y(f)$. A consequence of the Nonarchimedean Poincare-Lelong Formula, due to Thuillier, is the following.

Theorem 2.1 (Slope Formula)**.** *For any nonzero rational function $f \in K(C)$,*

$$\mathrm{Trop}(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f)).$$

For a divisor D on the metric graph Γ, we define

$$R(D) := \{\psi \in \mathrm{PL}(\Gamma) | \, \mathrm{div}(\psi) + D \geq 0\}.$$

The slope formula implies that, for a divisor D on a curve X, we have

$$\operatorname{trop}(\mathcal{L}(D)) \subseteq R(\operatorname{Trop}(D)).$$

We note that the containment here is often strict. We can now define a tropical notion of linear independence.

Definition 2.2. *A set* $\{\psi_1, \ldots, \psi_k\} \subset \mathrm{PL}(\Gamma)$ *is* tropically dependent *if there exist constants* $b_i \in \mathbb{R}$ *such that the minimum*

$$\min_i \{b_i + \psi_i\}$$

occurs at least twice at every point $v \in \Gamma$*. We say that the set is* tropically independent *otherwise.*

Tropical independence is a useful tool due to the following lemma.

Lemma 2.3. *If a set of rational functions* $\{f_1, \ldots, f_k\} \subset K(X)^*$ *is linearly dependent then* $\{\operatorname{trop}(f_1), \ldots, \operatorname{trop}(f_k)\} \subset \mathrm{PL}(\Gamma)$ *is tropically dependent.*

Hence, our strategy for proving that some collection of rational functions is linearly independent will be to specialize to the skeleton Γ, and then prove that the specializations are tropically independent. In many situations, one can prove the tropical independence of certain functions using the following basic fact about the shapes of divisors associated to a pointwise minimum of functions in a tropical linear series.

Shape Lemma for Minima. [JP14, Lemma 3.4] Let D be a divisor on a metric graph Γ, with $\psi_0, \ldots, \psi_r$ piecewise linear functions in $R(D)$, and let

$$\theta = \min\{\psi_0, \ldots, \psi_r\}.$$

Let $\Gamma_j \subset \Gamma$ be the closed set where θ is equal to ψ_j. Then $\operatorname{div}(\theta) + D$ contains a point $v \in \Gamma_j$ if and only if v is in either

(1) the divisor $\operatorname{div}(\psi_j) + D$, or
(2) the boundary of Γ_j.

3. The Gieseker-Petri Theorem

We now explain how to use tropical methods to prove the Gieseker-Petri Theorem. Recall that it suffices to exhibit a curve X such that the natural maps

$$\mu : \mathcal{L}(D) \otimes \mathcal{L}(K_X - D) \to \mathcal{L}(K_X)$$

are injective for all divisors D on X. For simplicity, we may assume that we are working over an algebraically closed field that is spherically complete with respect to a valuation that surjects onto the real numbers. Any metric graph Γ of genus g occurs as the skeleton of a smooth projective genus g curve X over such a field, hence we are free to choose any metric graph we like and assume that we have a curve X with skeleton Γ.

In [CDPR12], Cools, Draisma, Payne and Robeva consider the family of graphs pictured in Figure 1, colloquially known as the chain of loops. They assume the edge lengths to be generic, which in this case means that, if ℓ_i, m_i are the lengths of the bottom and top edges of the ith loop, then ℓ_i/m_i is not equal to the ratio of two positive integers whose sum is less than or equal to $2g - 2$.

To prove the Gieseker-Petri Theorem, we use the same metric graph. Given a divisor $D \in W^r_d(X)$, the goal is to find functions

$$\psi_0, \ldots, \psi_r \in \operatorname{trop}(\mathcal{L}(D)) \quad \text{and} \quad \varphi_0, \ldots, \varphi_{g-d+r-1} \in \operatorname{trop}(\mathcal{L}(K_X - D))$$

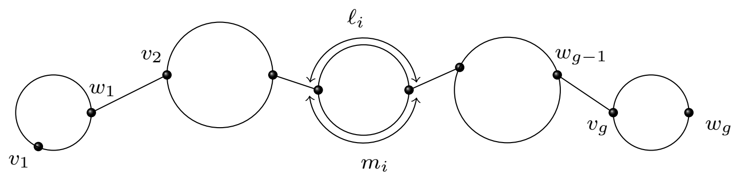

FIGURE 1. The graph Γ.

such that $\{\psi_i + \varphi_j\}_{i,j}$ is tropically independent.

There is a dense open subset of $W^r_d(\Gamma)$ consisting of divisors D with the following property: given an integer $0 \leq i \leq r$, there exists a unique divisor $D_i \sim D$ such that

$$D_i - iw_g - (r-i)v_1 \geq 0.$$

Following [CJP14], we refer to these divisors as *vertex avoiding*.

We first describe how to prove the injectivity of the map in the case that D is vertex avoiding. If D is the specialization of a divisor $\mathcal{D} \in W^r_d(C)$, and $p_1, p_g \in C$ are points specializing to v_1, w_g, respectively, then there exists a divisor $\mathcal{D}_i \sim \mathcal{D}$ such that

$$\mathcal{D}_i - ip_g - (r-i)p_1 \geq 0,$$

and, by the uniqueness of D_i, $\mathcal{D}_i$ must specialize to D_i. It follows that there is a function $\psi_i \in \mathrm{trop}(\mathcal{L}(\mathcal{D}))$ such that $\mathrm{div}(\psi_i) = D_i - D$, and similarly for $K_X - D$.

For this open subset of divisors, the argument then proceeds as follows. By the explicit description of $W^r_d(\Gamma)$ in [CDPR12], the divisor D_i fails to have a chip on the kth loop if and only if the integer k appears in the ith column of the corresponding tableau. The adjoint divisor $E = K_\Gamma - D$ corresponds to the transpose tableau [AMSW13, Theorem 39], so the divisor $D_i + E_j$ fails to have a chip on the kth loop if and only if k appears in the (i,j) position of the tableau. Since for each k at most one of these divisors fails to have a chip on the kth loop, we see that if

$$\theta = \min\{\psi_i + \varphi_j + b_{i,j}\}$$

occurs at least twice at every point of Γ, then the divisor

$$\Delta = \mathrm{div}(\theta) + K_\Gamma$$

must have a chip on the kth loop for all k.

To see that this is impossible, let p_k be a point of Δ in γ_k, and let

$$D' = p_1 + \cdots + p_g.$$

Then by construction $K_\Gamma - D'$ is equivalent to an effective divisor, so by the tropical Riemann-Roch Theorem we see that $r(D') \geq 1$. But the divisor D' is an example of a *simple break divisor*, as defined in [ABKS14], and all simple break divisors have rank 0, a contradiction.

It is interesting to note that this obstruction is, at heart, combinatorial. Unlike the earlier proofs via limit linear series, which arrive at a contradiction by constructing a canonical divisor of impossible *degree* (larger than $2g-2$), this argument arrives at a contradiction by constructing a canonical divisor of impossible *shape*.

The major obstacle to extending this argument to the case where D is not vertex avoiding is that the containment $\mathrm{trop}(\mathcal{L}(D)) \subseteq R(\mathrm{Trop}(D))$ is often strict. Given an arbitrary divisor $D \in W^r_d(X)$ and function $\psi \in R(\mathrm{Trop}(D))$, it is difficult to determine whether ψ is the specialization of a function in $\mathcal{L}(D)$. To avoid this issue, we make use of a patching construction, gluing together tropicalizations of different rational functions in a fixed algebraic linear series on different parts of the graph, to arrive at a piecewise linear function in $R(K_\Gamma)$ that may not be in $\mathrm{trop}(\mathcal{L}(K_X))$. Once this piecewise linear function is constructed, the argument proceeds very similarly to the vertex avoiding case.

3.1. **The Maximal Rank Conjecture.** We now turn to the Maximal Rank Conjecture. While this conjecture remains open in general, several important cases are known [BE85, Voi92, Tib03, Far09]. For example, it is shown in [BE85] that the Maximal Rank Conjecture holds in the non-special range $d \geq g + r$. When $d < g + r$, the general linear series of degree d and rank r on a general curve is complete, and for this reason, most of the work in the subject focuses on the case where $V = \mathcal{L}(D)$. It is interesting that, aside from [Tib03], for the most part these arguments do not make use of limit linear series.

In [JP], we use tropical Brill-Noether theory to prove the $m = 2$ case of the Maximal Rank Conjecture.

Theorem 3.1. [JP] *Let X be a smooth projective curve of genus g over a complete nonarchimedean field such that the minimal skeleton of the Berkovich analytic space X^{an} is isometric to a generic chain of loops Γ. For a given r and d, let D be a general divisor of rank r and degree d on X. Then the multiplication map*

$$\mu_2 : Sym^2\mathcal{L}(D) \to \mathcal{L}(2D)$$

has maximal rank.

The genericity conditions placed on the edge lengths of Γ in Theorem 3.1 are stricter than those appearing in the tropical proofs of the Brill-Noether and Gieseker-Petri Theorems. First, the bridges between the loops are assumed to be much longer than the loops themselves, and second, one must assume that certain integer linear combinations of the edge lengths do not vanish.

A simplifying aspect of the Maximal Rank Conjecture is that it concerns a general, rather than arbitrary, divisor. It therefore suffices to prove that the maximal rank condition holds for a single divisor of the given degree and rank on X. The main result of [CJP14] is that every divisor on the generic chain of loops is the specialization of a divisor of the same rank on X. We are therefore free to choose whatever divisor we wish to work with, and in particular we may choose one of the vertex avoiding divisors described in the previous section. Recall that, if $D \in W^r_d(\Gamma)$ is vertex avoiding, then we have an explicit set of piecewise linear functions $\psi_i \in R(D)$ that are tropicalizations of a basis for the linear series on the curve X. The goal, in the case where the multiplication map is supposed to be injective, is to show that the set $\{\psi_i + \psi_j\}_{i \leq j}$ is tropically independent. In the surjective case, we must choose a subset of the appropriate size, and then show that this subset is tropically independent.

The basic idea of the argument is as follows. Assume that

$$\theta = \min\{\psi_i + \psi_j + b_{i,j}\}$$

occurs at least twice at every point of Γ, and consider the divisor

$$\Delta = \mathrm{div}(\theta) + 2D.$$

To arrive at a contradiction, one studies the degree distribution of the divisor Δ across the loops of Γ. More precisely, one defines

$$\delta_k := \deg(\Delta|_{\gamma_k}).$$

The first step is to show that $\delta_k \geq 2$ for all k. One then identifies intervals $[a, b]$ for which this inequality must be strict for at least one $k \in [a, b]$. As one proceeds from left to right across the graph, one encounters such intervals sufficiently many times to obtain $\deg \Delta > 2 \deg D$, a contradiction.

References

[ABKS14] Y. An, M. Baker, G. Kuperberg, and F. Shokrieh. Canonical representatives for divisor classes on tropical curves and the matrix-tree theorem. *Forum Math. Sigma*, 2:e24, 25, 2014.

[AMSW13] R. Agrawal, G. Musiker, V. Sotirov, and F. Wei. Involutions on standard Young tableaux and divisors on metric graphs. *Electron. J. Combin.*, 20(3), 2013.

[BE85] E. Ballico and Ph. Ellia. The maximal rank conjecture for nonspecial curves in $\mathbf{P}^3$. *Invent. Math.*, 79(3):541–555, 1985.

[CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.*, 230(2):759–776, 2012.

[CJP14] D. Cartwright, D. Jensen, and S. Payne. Lifting divisors on a generic chain of loops. preprint arXiv:1404.4001. To appear in *Canad. Math. Bull.*, 2014.

[Far09] G. Farkas. Koszul divisors on moduli spaces of curves. *Amer. J. Math.*, 131(3):819–867, 2009.

[Gie82] D. Gieseker. Stable curves and special divisors: Petri's conjecture. *Invent. Math.*, 66(2):251–275, 1982.

[JP] D. Jensen and S. Payne. Tropical independence II: The Maximal Rank Conjecture for quadrics. in preparation.

[JP14] D. Jensen and S. Payne. Tropical independence I: Shapes of divisors and a proof of the Gieseker-Petri theorem. *Algebra Number Theory*, 8(9):2043–2066, 2014.

[Tib03] Montserrat Teixidor i bigas. Injectivity of the symmetric map for line bundles. *Manuscripta Math.*, 112(4):511–517, 2003.

[Voi92] Claire Voisin. Sur l'application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri. *Acta Math.*, 168(3-4):249–272, 1992.