CONVERGENCE POLYGONS FOR CONNECTIONS ON NONARCHIMEDEAN CURVES

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In classical analysis, one builds the catalog of special functions by repeatedly adjoining solutions of differential equations whose coefficients are previously known functions. Consequently, the properties of special functions depend crucially on the basic properties of ordinary differential equations. This naturally led to the study of formal differential equations, as in the seminal work of Turrittin [46]; this may be viewed retroactively as a theory of differential equations over a trivially valued field. After the introduction of p-adic analysis in the early 20th century, there began to be corresponding interest in solutions of p-adic differential equations; however, aside from some isolated instances (e.g., the proof of the Nagell-Lutz theorem; see Theorem 3.4), a unified theory of p-adic ordinary differential equations did not emerge until the pioneering work of Dwork on the relationship between p-adic special functions and the zeta functions of algebraic varieties over finite fields (e.g., see [15, 16]). At that point, serious attention began to be devoted to a serious discrepancy between the p-adic and complex-analytic theories: on an open p-adic disc, a nonsingular differential equation can have a formal solution which does not converge in the entire disc (e.g., the exponential series). One is thus led to quantify the convergence of power series solutions of differential equations involving rational functions over a nonarchimedean field; this was originally done by Dwork in terms of the *generic radius of convergence* [17]. This and more refined invariants were studied by numerous authors during the half-century following Dwork's initial work, as documented in the author's book [25].

At around the time that [25] was published, a new perspective was introduced by Baldassarri [3] (and partly anticipated in prior unpublished work of Baldassarri and Di Vizio [4]) which makes full use of Berkovich's theory of nonarchimedean analytic spaces. Given a differential equation as above, or more generally a connection on a curve over a nonarchimedean field, one can define an invariant called the *convergence polygon*; this is a function from the underlying Berkovich topological space of the curve into a space of Newton polygons, which measures the convergence of formal horizontal sections and is well-behaved with respect to both the topology and the piecewise linear structure on the Berkovich space. One can translate much of the prior theory of p-adic differential equations into (deceptively) simple statements about the behavior of the convergence polygon; this process was carried out in a series of papers by Poineau and Pulita [37, 33], as supplemented by work of this author [28] and upcoming joint work with Baldassarri [5].

Date: February 6, 2015.

Thanks to Francesco Baldassarri, Andrea Pulita, Junecue Suh, and the participants of the Simons Symposium for feedback. The author was supported by NSF grant DMS-1101343 and UC San Diego (Warschawski Professorship), and by a Simons Visiting Professorship grant during November 2014.

In this paper, we present the basic theorems on the convergence polygon, which provide a number of combinatorial constraints that may be used to extract information about convergence of formal horizontal sections at one point from corresponding information at other points. We include numerous examples to illustrate some typical behaviors of the convergence polygon. We also indicate some relationships between convergence polygons and the geometry of finite morphisms, paying special attention to the case of cyclic p-power coverings with p equal to the residual characteristic. This case is closely linked with the Oort lifting problem for Galois covers of curves in characteristic p, and some combinatorial constructions arising in that problem turn out to be closely related to convergence polygons. There are additional applications to the study of integrable connections on higher-dimensional nonarchimedean analytic spaces, both in the cases of zero residual characteristic [26] and positive residual characteristic [27], but we do not pursue these applications here.

To streamline the exposition, we make no attempt to indicate the techniques of proof underlying our main results; in some cases, quite sophisticated arguments are required. We limit ourselves to saying that the basic tools are developed in a self-contained fashion in [25], and the other aforementioned results are obtained by combining results from [25] in an intricate manner. (Two exceptions are made for results which do not occur in any existing paper; their proofs are relegated to appendices.) We also restrict generality by considering only proper curves, even though many of the results we discuss can be formulated for open curves, possibly of infinite genus.

1. Newton polygons

As setup for our definition of convergence polygons, we fix some conventions regarding Newton polygons.

Definition 1.1. For n a positive integer, let $\mathcal{P}[0,n]$ be the set of continuous functions $\mathcal{N} : [0,n] \to \mathbb{R}$ satisfying the following conditions.

(a) We have $\mathcal{N}(0) = 0$.
(b) For $i = 1, \dots, n$, the restriction of $\mathcal{N}$ to $[i-1, i]$ is affine.

For $i = 1, \dots, n$, we write $h_i : \mathcal{P}[0,n] \to \mathbb{R}$ for the function $\mathcal{N} \mapsto \mathcal{N}(i)$; we call $h_i(\mathcal{N})$ the i-th *height* of $\mathcal{N}$. The product map $h_1 \times \cdots \times h_n : \mathcal{P}[0,n] \to \mathbb{R}^n$ is a bijection, using which we equip $\mathcal{P}[0,n]$ with a topology and an integral piecewise linear structure. We sometimes refer to h_n simply as h and call it the *total height*.

Definition 1.2. Let $\mathcal{NP}[0,n]$ be the subset of $\mathcal{P}[0,n]$ consisting of *concave* functions. For $i = 1, \dots, n$, we write $s_i : \mathcal{P}[0,n] \to \mathbb{R}$ for the function $\mathcal{N} \mapsto \mathcal{N}(i) - \mathcal{N}(i-1)$; we call $s_i(\mathcal{N})$ the i-th *slope* of $\mathcal{N}$. For $\mathcal{N} \in \mathcal{NP}[0,n]$, we have $s_1(\mathcal{N}) \geq \cdots \geq s_n(\mathcal{N})$.

Definition 1.3. Let $I \subseteq \mathbb{R}$ be a closed interval. A function $\mathcal{N} : I \to \mathcal{NP}[0,n]$ is *affine* if it has the form $\mathcal{N}(t) = \mathcal{N}_0 + t\mathcal{N}_1$ for some $\mathcal{N}_0, \mathcal{N}_1 \in \mathcal{P}[0,n]$. In this case, we call $\mathcal{N}_1$ the *slope* of $\mathcal{N}$. We say that $\mathcal{N}$ has *integral derivative* if $\mathcal{N}_1(i) \in \mathbb{Z}$ for $i = n$ and for each $i \in \{1, \dots, n-1\}$ such that for all t in the interior of I, $\mathcal{N}(t)$ has a change of slope at i. This implies that the graph of $\mathcal{N}_1$ has vertices only at lattice points, but not conversely. (It would be natural to use the terminology *integral slope*, but we avoid this terminology to alleviate confusion with Definition 1.2.)

2. PL structures on Berkovich curves

We next recall the canonical piecewise linear structure on a Berkovich curve (e.g., see [6]).

Hypothesis 2.1. For the rest of this paper, let K be a *nonarchimedean field*, i.e., a field complete with respect to a nonarchimedean absolute value; let X be a smooth, proper, geometrically connected curve over K; let Z be a finite set of closed points of X; and put $U = X - Z$.

Convention 2.2. Whenever we view $\mathbb{Q}_p$ as a nonarchimedean field, we normalize the p-adic absolute value so that $|p| = p^{-1}$.

Remark 2.3. Recall that the points of the Berkovich analytification X^{an} may be identified with equivalence classes of pairs (L, x) in which L is a nonarchimedean field over K and x is an element of $X(L)$, where the equivalence relation is generated by relations of the form $(L, x) \sim (L', x')$ where x' is the restriction of x along a continuous K-algebra homomorphism $L \to L'$. As is customary, we classify points of X^{an} into types 1,2,3,4 (e.g., see [28, Proposition 4.2.7]). To lighten notation, we identify Z with Z^{an}, which is a finite subset of X^{an} consisting of type 1 points.

Definition 2.4. For $\rho > 0$, let x_ρ denote the generic point of the disc $|z| \leq \rho$ in $\mathbb{P}^1_K$. A *segment* in X^{an} is a closed subspace S homeomorphic to a closed interval for which there exist an open subspace U of X^{an}, a choice of values $0 \leq \alpha < \beta \leq +\infty$, and an isomorphism of U with $\{z \in \mathbb{P}^{1,\mathrm{an}}_K : \alpha < |z| < \beta\}$ identifying the interior of S with $\{x_\rho : \rho \in (\alpha, \beta)\}$. A *virtual segment* in X^{an} is a connected closed subspace whose base extension to some finite extension of K is a disjoint union of segments.

A *strict skeleton* in X^{an} is a subspace Γ containing Z^{an} equipped with a homeomorphism to a finite connected graph, such that each vertex of the graph corresponds to either a point of Z or a point of type 2, and each edge corresponds to a virtual segment, and X^{an} retracts continuously onto Γ. Using either tropicalizations or semistable models, one may realize X^{an} as the inverse limit of its strict skeleta; again, see [6] for a detailed discussion.

Definition 2.5. Note that

$$\chi(U) = 2 - 2g(X) - \mathrm{length}(Z),$$

so $\chi(U) \leq 0$ if and only if either $g(X) \geq 1$ or $\mathrm{length}(Z) \geq 2$. In this case, there is a unique minimal strict skeleton in X^{an}, which we denote $\Gamma_{X,Z}$. Explicitly, if K is algebraically closed, then the underlying set of $\Gamma_{X,Z}$ is the complement in X^{an} of the union of all open discs in U^{an}; for general K, the underlying set of $\Gamma_{X,Z}$ is the image under restriction of the minimal strict skelelon in X^{an}_L for L a completed algebraic closure of K. Beware that $\Gamma_{X,Z}$ is not invariant under base extension from K to L if L is not the completion of an algebraic extension of K; see Definition 8.4 for a related phenomenon.

3. Convergence polygons: projective space

We next introduce the concept of the convergence polygon associated to a differential equation on $\mathbb{P}^1$.

Hypothesis 3.1. For the rest of this paper, we assume that the nonarchimedean field K is of characteristic 0, as otherwise the study of differential operators on K-algebras has a markedly different flavor (for instance, any derivation on a ring R of characteristic $p > 0$ has the subring of p-th powers in its kernel). By contrast, the residue characteristic of K, which we call p, may be either 0 or positive unless otherwise specified (e.g., if we refer to $\mathbb{Q}_p$ then we implicitly require $p > 0$).

Hypothesis 3.2. For the rest of §3, take $X = \mathbb{P}^1_K$ and consider the differential equation

$$y^{(n)} + f_{n-1}(z)y^{(n-1)} + \cdots + f_0(z)y = 0 \tag{3.2.1}$$

for some rational functions $f_0, \dots, f_{n-1} \in K(z)$ with poles only within Z. If $Z = \{\infty\}$, let m be the dimension of the K-vector space of entire solutions of (3.2.1); otherwise, take $m = 0$.

Definition 3.3. For any nonarchimedean field L over K and any $x \in U(L)$, let S_x be the set of formal solutions of (3.2.1) with $y \in L[\![z-x]\!]$. By interpreting (3.2.1) as a linear recurrence relation of order n on the coefficients of a power series, we see that every list of n initial conditions at $z = x$ corresponds to a unique formal solution; that is, the composition

$$S_x \to L[\![z-x]\!] \to L[\![z-x]\!]/(z-x)^n$$

is a bijection. In particular, S_x is an L-vector space of dimension n.

Theorem 3.4 (p-adic Cauchy theorem)**.** *Each element of S_x has a positive radius of convergence.*

Proof. This result was originally proved by Lutz [29, Théorème IV] somewhat before the emergence of the general theory of p-adic differential equations; Lutz used it as a lemma in her proof of the *Nagell-Lutz theorem* on the integrality of torsion points on rational elliptic curves. One can give several independent proofs using the modern theory; see [25, Proposition 9.3.3, Proposition 18.1.1]. □

Definition 3.5. For $i = 1, \dots, n-m$, choose $s_i(x) \in \mathbb{R}$ so that $e^{-s_i(x)}$ is the supremum of the set of $\rho > 0$ such that U^{an} contains the open disc $|z-x| < \rho$ and S_x contains $n-i+1$ linearly independent elements convergent on this disc. Note that this set is nonempty by Theorem 3.4 and bounded above by the definition of m, so the definition makes sense. In particular, $s_1(x)$ is the joint radius of convergence of all of the elements of S_x, while $s_{n-m}(x)$ is the maximum finite radius of convergence of a nonzero element of S_x.

Since $s_1(x) \geq \cdots \geq s_{n-m}(x)$, the $s_i(x)$ are the slopes of a polygon $\mathcal{N}_z(x) \in \mathcal{NP}[0, n-m]$, which we call the *convergence polygon* of (3.2.1) at x. (We include z in the notation to remind ourselves that $\mathcal{N}_z$ depends on the choice of the coordinate z of X.) This construction is compatible with base change: if L' is a nonarchimedean field containing L and x' is the image of x in $U(L')$, then $\mathcal{N}_z(x) = \mathcal{N}_z(x')$. Consequently, we obtain a well-defined function $\mathcal{N}_z : U^{\mathrm{an}} \to \mathcal{NP}[0, n-m]$.

Definition 3.6. By definition, $e^{-s_1(\mathcal{N}_z(x))}$ can never exceed the largest value of ρ for which the disc $|z-x| < \rho$ does not meet Z. When equality occurs, we say that (3.2.1) satisfies the *Robba condition* at x.

Theorem 3.7. *The function $\mathcal{N}_z : U^{\mathrm{an}} \to \mathcal{NP}[0, n-m]$ is continuous; more precisely, it factors through the retraction of $\mathbb{P}^{1,\mathrm{an}}_K$ onto some strict skeleton Γ, and the restriction of $\mathcal{N}_z$ to each edge of Γ is affine with integral derivative.*

Proof. See [37] or [28] or [5]. □

One can say quite a bit more, but for this it is easier to shift to a coordinate-free interpretation, which also works for more general curves; see §5.

4. A gallery of examples

To help the reader develop some intuition, we collect a few illustrative examples of convergence polygons. Throughout §4, retain Hypothesis 3.1.

Example 4.1. Take $K = \mathbb{Q}_p$, $Z = \{\infty\}$, and consider the differential equation $y' - y = 0$. The formal solutions of this equation with $y \in L[\![z - x]\!]$ are the scalar multiples of the exponential series

$$\exp(z - x) = \sum_{i=0}^{\infty} \frac{(z - x)^i}{i!},$$

which has radius of convergence $p^{-1/(p-1)}$. Consequently,

$$s_1(\mathcal{N}_z(x)) = \frac{1}{p-1} \log p;$$

in particular, $\mathcal{N}_z$ is constant on U^{an}.

In this next example, we illustrate the effect of changing Z on the convergence polygon.

Example 4.2. Set notation as in Example 4.1, except now with $Z = \{0, \infty\}$. In this case we have

$$s_1(\mathcal{N}_z(x)) = \max \left\{ -\log |x|, \frac{1}{(p-1)} \log p \right\}.$$

In particular, $\mathcal{N}_z$ factors through the retraction of $\mathbb{P}^{1,\mathrm{an}}_K$ onto the path from 0 to ∞. For $x \in U^{\mathrm{an}}$, the Robba condition holds at x if and only if $|x| \geq p^{-1/(p-1)}$.

Example 4.3. Take $K = \mathbb{Q}_p$, $Z = \{0, \infty\}$, and consider the differential equation $y' - \frac{1}{p} z^{-1} y = 0$. The formal solutions of this equation with $y \in L[\![z - x]\!]$ are the scalar multiples of the binomial series

$$\sum_{i=0}^{\infty} \binom{1/p}{i} x^{1/p - i} (z - x)^i,$$

which has radius of convergence $p^{-p/(p-1)}|x|$. Consequently,

$$s_1(\mathcal{N}_z(x)) = \frac{p}{p-1} \log p - \log |x|,$$

so again $\mathcal{N}_z$ factors through the retraction of $\mathbb{P}^{1,\mathrm{an}}_K$ onto the path from 0 to ∞. In this case, the Robba condition holds nowhere.

Example 4.4. Assume $p > 2$, take $K = \mathbb{Q}_p$, $Z = \{0, \infty\}$, and consider the Bessel differential equation (with parameter 0)

$$y'' + z^{-1} y' + y = 0.$$

This example was studied by Dwork [18], who showed that

$$s_1(\mathcal{N}_z(x)) = s_2(\mathcal{N}_z(x)) = \max \left\{ -\log |x|, \frac{1}{p-1} \log p \right\}.$$

Again, $\mathcal{N}_z$ factors through the retraction of $\mathbb{P}^{1,\mathrm{an}}_K$ onto the path from 0 to ∞. As in Example 4.2, for $x \in U^{\mathrm{an}}$, the Robba condition holds at x if and only if $|x| \geq p^{-1/(p-1)}$.

Our next example illustrates a typical effect of varying a parameter.

Example 4.5. Let K be an extension of $\mathbb{Q}_p$, take $Z = \{0, \infty\}$, and consider the differential equation $y' - \lambda z^{-1} y = 0$ for some $\lambda \in K$ (the case $\lambda = 1/p$ being Example 4.3). Then

$$s_1(\mathcal{N}_z(x)) = c + \frac{1}{p-1} \log p - \log |x|$$

where c is a continuous function of

$$c_0 = \min\{|\lambda - t| : t \in \mathbb{Z}_p\};$$

namely, we have

$$c = \begin{cases} \log |c_0| & c_0 \geq 1 \\ -\frac{p^m - 1}{(p-1)p^m} \log p + \frac{1}{p^m} \log |c_0| & p^{-m} \leq c_0 \leq p^{-m+1}, m = 1, 2, \dots \\ -\frac{1}{p-1} \log p & c_0 = 0. \end{cases}$$

In particular, the Robba condition holds everywhere if $\lambda \in \mathbb{Z}_p$ and nowhere otherwise. In either case, $\mathcal{N}_z$ factors through the retraction of $\mathbb{P}^{1,\mathrm{an}}_K$ onto the path from 0 to ∞.

Example 4.6. Take $K = \mathbb{Q}_p$, $Z = \{\infty\}$, and consider the differential equation $y' - az^{a-1} y = 0$ for some positive integer a not divisible by p (the case $a = 1$ being Example 4.1). The formal solutions of this equation are the scalar multiples of

$$\exp(z^a - x^a).$$

This series converges in the region where $|z^a - x^a| < p^{-1/(p-1)}$; consequently,

$$s_1(\mathcal{N}_z(x)) = \max\left\{ \frac{1}{p-1} \log p + (a-1) \log |x|, \frac{1}{a(p-1)} \log p \right\}.$$

In this case, $\mathcal{N}_z$ factors through the retraction onto the path from $x_{p^{-1/(p-1)}}$ to ∞.

Example 4.7. Take $K = \mathbb{C}((t))$ (so $p = 0$), $Z = \{\infty\}$, and consider the differential equation

$$y''' + zy'' + y = 0.$$

It can be shown that

$$s_1(\mathcal{N}_z(x)) = \max\{0, \log |x|\}, s_2(\mathcal{N}_z(x)) = s_3(\mathcal{N}_z(x)) = \min\left\{0, -\frac{1}{2} \log |x|\right\}.$$

In this case, $\mathcal{N}_z$ factors through the retraction onto the path from x_1 to ∞. Note that this provides an example where the slopes of $\mathcal{N}_z(x)$ are not bounded below uniformly on $(\mathbb{P}^1_K - Z)^{\mathrm{an}}$; that is, as x approaches ∞, one local horizontal section has radius of convergence growing without bound, but these local horizontal sections do not patch together.

Example 4.8. Assume $p > 2$, take $K = \mathbb{Q}_p$, $Z = \{0, 1, \infty\}$, and consider the Gaussian hypergeometric differential equation

$$y'' + \frac{(1-2z)}{z(1-z)} y' - \frac{1}{4z(1-z)} y = 0.$$

This example was originally studied by Dwork [16] due to its relationship with the zeta functions of elliptic curves. Using Dwork's calculations, it can be shown that

$$s_1(\mathcal{N}_z(x)) = s_2(\mathcal{N}_z(x)) = \max\{\log |x|, -\log |x|, -\log |x - 1|\}.$$

In this case, $\mathcal{N}_z$ factors through the retraction from $\mathbb{P}^{1,\mathrm{an}}_K$ onto the union of the paths from 0 to ∞ and from 1 to ∞, and the Robba condition holds everywhere.

Remark 4.9. One can compute additional examples of convergence polygons associated to first-order differential equations using an explicit formula for the radius of convergence at a point, due to Christol–Pulita. This result was originally reported in [8] but with an error in the formula; for a corrected statement, see [38, Introduction, Théorème 5].

5. Convergence polygons: general curves

We now describe an analogue of the convergence polygon in a more geometric setting.

Hypothesis 5.1. Throughout §5, assume that $\chi(U) \leq 0$, i.e., either $g(X) \geq 1$ or $\mathrm{length}(Z) \geq 2$. Let $\mathcal{E}$ be a vector bundle on U of rank n equipped with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_U} \Omega_{U/K}$.

Remark 5.2. For the results in this section, one could also allow X to be an analytic curve which is compact but not necessarily proper. To simplify the discussion, we omit this level of generality.

Definition 5.3. Let L be a nonarchimedean field and choose $x \in U(L)$. Since X is smooth, U_L^{an} contains a neighborhood of x isomorphic to an open disc over L. Thanks to our restrictions on X and Z, the union U_x of all such neighborhoods is itself isomorphic to an open disc over L. For each $\rho \in (0,1]$, let $U_{x,\rho}$ be the open disc of radius ρ centered at x within U_x (normalized so that $U_{x,1} = U_x$).

Let $\widehat{\mathcal{O}}_{X,x}$ denote the completed local ring of X at x; it is abstractly a power series ring in one variable over L. Let $\mathcal{E}_x$ denote the pullback of $\mathcal{E}$ to $\widehat{\mathcal{O}}_{X,x}$, equipped with the induced connection. One checks easily that $\mathcal{E}_x$ is a trivial differential module; more precisely, the space $\ker(\nabla, \mathcal{E}_x)$ is an n-dimensional vector space over L and the natural map

$$\ker(\nabla, \mathcal{E}_x) \otimes_L \widehat{\mathcal{O}}_{X,x} \to \mathcal{E}_x$$

is an isomorphism.

For $i = 1, \dots, n$, choose $s_i(x) \in [0, +\infty)$ so that $e^{-s_i(x)}$ is the supremum of the set of $\rho \in (0,1]$ such that $\mathcal{E}_x$ contains $n - i + 1$ linearly independent sections convergent on $U_{x,\rho}$. Again, this set of such ρ is nonempty by Theorem 3.4. Since $s_1(x) \geq \cdots \geq s_n(x)$, the $s_i(x)$ are the slopes of a polygon $\mathcal{N}(x) \in \mathcal{NP}[0,n]$, which we call the *convergence polygon* of $\mathcal{E}$ at x. Again, the construction is compatible with base change, so it induces a well-defined function $\mathcal{N} : U^{\mathrm{an}} \to \mathcal{NP}[0,n]$.

Definition 5.4. For $x \in U^{\mathrm{an}}$, we say that $\mathcal{E}$ satisfies the *Robba condition* at x if $\mathcal{N}(x)$ is the zero polygon.

We have the following analogue of Theorem 3.7.

Theorem 5.5. *The function $\mathcal{N} : (X - Z)^{\mathrm{an}} \to \mathcal{NP}[0,n]$ is continuous. More precisely, there exists a strict skeleton Γ such that $\mathcal{N}$ factors through the retraction of X^{an} onto Γ, and the restriction of $\mathcal{N}$ to each edge of Γ is affine with integral derivative.*

Proof. See [33] or [28] or [5] (and Remark 5.6). □

Remark 5.6. It is slightly inaccurate to attribute Theorem 5.5 to [33] or [5], as the results proved therein are slightly weaker: they require an uncontrolled base extension on K, which creates more options for the strict skeleton Γ. In particular, Theorem 5.5 as stated implies that $\mathcal{N}$ is locally constant around any point of type 4, which cannot be established using the methods of [33] or [5]; one instead requires some dedicated arguments found only in [28]. These extra arguments are crucial for the applications of Theorem 5.5 in the contexts described in [26] and [27].

Remark 5.7. Suppose that $X = \mathbb{P}^1_K$ and $\infty \in Z$. Given a differential equation as in (3.2.1), we can construct an associated connection $\mathcal{E}$ of rank n whose underlying vector bundle is free on the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ and whose action of ∇ is given by

$$\begin{aligned}
\nabla(\mathbf{e}_1) &= f_0(z)\mathbf{e}_n \\
\nabla(\mathbf{e}_2) &= f_1(z)\mathbf{e}_n - \mathbf{e}_1 \\
&\vdots \\
\nabla(\mathbf{e}_{n-1}) &= f_{n-2}(z)\mathbf{e}_n - \mathbf{e}_{n-2} \\
\nabla(\mathbf{e}_n) &= f_{n-1}(z)\mathbf{e}_n - \mathbf{e}_{n-1}.
\end{aligned}$$

A section of $\mathcal{E}$ is then horizontal if and only if it has the form $y\mathbf{e}_1 + y'\mathbf{e}_2 + \cdots + y^{(n-1)}\mathbf{e}_n$ where y is a solution of (3.2.1). If $\mathrm{length}(Z) \geq 2$, each of $\mathcal{N}_z$ and $\mathcal{N}$ can be computed in terms of the other; this amounts to changing the normalization of certain discs. In particular, the statements of Theorem 3.7 and Theorem 5.5 in this case are equivalent.

If $\mathrm{length}(Z) = 1$, we cannot define $\mathcal{N}$ as above. However, if K is nontrivially valued, one can recover the properties of $\mathcal{N}_z$ by considering $\mathcal{N}$ with Z replaced by $Z \cup \{x\}$ for some $x \in U(K)$ with $|x|$ sufficiently large (namely, larger than the radius of convergence of any nonentire formal solution at 0). We refer to [5] for further details.

Remark 5.8. One can extend Remark 5.7 by defining $\mathcal{N}_z$ in the case where $X = \mathbb{P}^1_K$ and $\infty \in Z$, and using Theorem 5.5 to establish an analogue of Theorem 3.7. With this modification, we still do not define either $\mathcal{N}$ or $\mathcal{N}_z$ in the case where $X = \mathbb{P}^1_K$ and $Z = \emptyset$, but this case is completely trivial: the vector bundle $\mathcal{E}$ must admit a basis of horizontal sections (see [5]).

Theorem 5.9. *Suppose that $x \in \Gamma \cap U^{\mathrm{an}}$ is the generic point of a open disc D contained in X and the Robba condition holds at x.*

(a) *If $D \cap Z = \emptyset$, then the restriction of $\mathcal{E}$ to D is trivial (i.e., it admits a basis of horizontal sections).*
(b) *If $D \cap Z$ consists of a single point z at which ∇ is regular, then the Robba condition holds on $D - \{z\}$.*

Proof. Part (a) is a special case of the Dwork transfer theorem; see for instance [25, Theorem 9.6.1]. Part (b) follows as in the proof of [25, Theorem 13.7.1]. □

Remark 5.10. Theorem 5.9(b) is a variant of a result of Christol, which has a slightly stronger hypothesis and a slightly stronger conclusion. In Christol's result, one must assume either that $p = 0$, or that $p > 0$ and the pairwise differences between the exponents of ∇ at z are not p-adic Liouville numbers (see Example 7.19). One however gets the stronger conclusion that the "formal solution matrix" of ∇ at z converges on all of D.

Remark 5.11. Let k be the residue field of K. Suppose that $X = \mathbb{P}^1_K$, ∇ is regular everywhere, and the reduction map from Z to $\mathbb{P}^1_k$ is injective. Then Theorem 5.9 implies that if the Robba condition holds at x_1, then it holds on all of U^{an}. For instance, this is the case for (the connection associated via Remark 5.7 to) the hypergeometric equation considered in Example 4.8; more generally, it holds for the hypergeometric equation

$$y'' - \frac{c-(a+b+1)z}{z(1-z)}y' - \frac{ab}{z(1-z)}y = 0$$

if and only if $a, b, c \in \mathbb{Z}_p$ (the case of Example 4.8 being $a = b = 1/2, c = 1$). This example and Example 4.5, taken together, suggest that for a general differential equation with one or more accessory parameters, the Robba condition at a fixed point is likely to be of a "fractal" nature in these parameters. For some additional examples with four singular points, see the work of Beukers [7].

Remark 5.12. One can also consider some modified versions of the convergence polygon. For instance, one might take $e^{-s_i(x)}$ to be the supremum of those $\rho \in (0, 1]$ such that the restriction of $\mathcal{E}$ to $U_{x,\rho}$ splits off a trivial submodule of rank at least $n - i + 1$; the resulting convergence polygons will again satisfy Theorem 5.5. It may be that some modification of this kind can be used to eliminate some hypotheses on p-adic exponents, as in Theorem 7.23.

6. Derivatives of convergence polygons

We now take a closer look at the local variation of convergence polygons. Throughout §6, continue to retain Hypothesis 5.1.

Definition 6.1. For $x \in X^{\mathrm{an}}$, a *branch* of X at x is a local connected component of $X - \{x\}$, that is, an element of the direct limit of $\pi_0(U - \{x\})$ as U runs over all neighborhoods of x in X. Depending on the type of x, the branches of X can be described as follows.

1. A single branch.
2. One branch corresponding to each closed point on the curve C_x (defined over the residue field of K) whose function field is the residue field of $\mathcal{H}(x)$.
3. Two branches.
4. One branch.

For each branch $\vec{t}$ of X at x, by Theorem 5.5 we may define the *derivative* of $\mathcal{N}$ along $\vec{t}$ (away from x), as an element of $\mathcal{P}[0, n]$ with integral vertices; we denote this element by $\partial_{\vec{t}}(\mathcal{N})$. For x of type 1, we also denote this element by $\partial_x(\mathcal{N})$ since there is no ambiguity about the choice of the branch. We may similarly define $\partial_{\vec{t}}(h_i(\mathcal{N})) \in \mathbb{R}$ for $i = 1, \ldots, n$, optionally omitting i in the case $i = n$; note that $\partial_{\vec{t}}(h(\mathcal{N})) \in \mathbb{Z}$.

Theorem 6.2. *For $z \in Z$, $-\partial_z(\mathcal{N})$ is the polygon associated to the Turrittin-Levelt-Hukuhara decomposition of $\mathcal{E}_z$ (see for example* [25, Chapter 7]*). In particular, this polygon belongs to $\mathcal{NP}[0, n]$, its slopes are all nonnegative, and its height equals the irregularity* $\mathrm{Irr}_z(\nabla)$ *of ∇ at z.*

Proof. See [5]. □

Corollary 6.3. *For $z \in Z$, $\mathcal{N}$ extends continuously to a neighborhood of z if and only if ∇ has a regular singularity at z (i.e., its irregularity at z equals 0). In particular, $\mathcal{N}$ extends continuously to all of X^{an} if and only if ∇ is everywhere regular.*

Remark 6.4. Using a similar technique, one can compute the asymptotic behavior of $\mathcal{N}$ in a neighborhood of $z \in Z$ in terms of the "eigenvalues" occurring in the Turrittin-Levelt-Hukuhara decomposition of $\mathcal{E}_z$. For example, ∇ satisfies the Robba condition on some neighborhood of z if and only if ∇ is regular at z with all exponents in $\mathbb{Z}_p$.

Theorem 6.5. *For $x \in U^{\mathrm{an}}$ and $\vec{t}$ a branch of X at x not pointing along $\Gamma_{X,Z}$, we have $\partial_{\vec{t}}(\mathcal{N}) \leq 0$.*

Proof. This requires somewhat technical arguments not present in the existing literature; see Theorem A.9 for the case $p > 0$ and Theorem B.4 for the case $p = 0$. □

Remark 6.6. In the setting of Theorem 6.5, the statement that $\partial_{\vec{t}}(h_1(\mathcal{N})) \leq 0$ is equivalent to the Dwork transfer theorem (again see [25, Theorem 9.6.1]). For $p = 0$, Theorem 6.5 is deduced by relating $\partial_{\vec{t}}(\mathcal{N})$ to local indices, as discussed in §7; for $p > 0$, one uses a suitable perturbation to reduce to the case where $\mathcal{N}(x)$ has no slopes equal to 0, to which results of [25] may be applied. See Appendix A for details.

Remark 6.7. By Theorem 5.5, for each $x \in U^{\mathrm{an}}$, there exist only finitely many branches $\vec{t}$ at x along which $\mathcal{N}$ has nonzero slope. If x is of type 1 or 4, there are in fact no such branches. If x is of type 3, then the slopes along the two branches at x add up to 0.

7. Subharmonicity and index

Using the piecewise affine structure of the convergence polygon, we formulate some additional properties, including local and global index formulas for de Rham cohomology. The local index formula is due to Poineau and Pulita [34], generalizing some partial results due to Robba [39, 40, 41, 42] and Christol-Mebkhout [9, 10, 11, 12]. Unfortunately, in the case $p > 0$ one is forced to interact with a fundamental pathology in the theory of p-adic differential equations, namely the effect of *p-adic Liouville numbers*; consequently, the global formula we derive here cannot be directly deduced from the local formula (see Remark 7.14 and Remark 7.18).

Hypothesis 7.1. Throughout §7, continue to retain Hypothesis 5.1, but assume in addition that K is algebraically closed. (Without this assumption, one can still formulate the results at the expense of having to keep track of some additional multiplicity factors.)

Definition 7.2. For $x \in U^{\mathrm{an}}$, let $(\Delta\mathcal{N})_x \in \mathcal{P}[0,n]$ denote the sum of $\partial_{\vec{t}}(\mathcal{N})$ over all branches $\vec{t}$ of X at x; by Remark 6.7, this sum can only be nonzero when x is of type 2. Define the *Laplacian* of $\mathcal{N}$ as the $\mathcal{P}[0,n]$-valued measure $\Delta\mathcal{N}$ taking a continuous function $f : U^{\mathrm{an}} \to \mathbb{R}$ to $\sum_{x \in U^{\mathrm{an}}} f(x)(\Delta\mathcal{N})_x$. For $i = 1, \dots, n$, we may similarly define the multiplicities $(\Delta h_i(\mathcal{N}))_x \in \mathbb{R}$ and the Laplacian $\Delta h_i(\mathcal{N})$; we again omit the index i when it equals n.

Remark 7.3. The definition of the Laplacian can also be interpreted in the context of Thuillier's potential theory [45], which applies more generally to functions which need not be piecewise affine.

Lemma 7.4. *We have*

$$\int \Delta h(\mathcal{N}) = \sum_{z \in Z} \mathrm{Irr}_z(\nabla).$$

Proof. For e an edge of Γ, we may compute the slopes of $\mathcal{N}$ along the two branches pointing into e from the endpoints of e; these two slopes add up to 0. If we add up these slopes over all e, then regroup this sum by vertices, then the sum at each vertex $z \in Z$ equals $-\operatorname{Irr}_z(\nabla)$ by Theorem 6.2, while the sum at each vertex $x \in U^{\mathrm{an}}$ is the multiplicity of x in $\Delta\mathcal{N}$. This proves the claim. $\square$

Definition 7.5. For any open subset V of X^{an}, consider the complex

$$0 \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega \to 0$$

of sheaves, keeping in mind that if $V \cap Z \neq \emptyset$, then the sections over V are allowed to be meromorphic at $V \cap Z$ (but not to have essential singularities; see Remark 7.8). We define $\chi_{\mathrm{dR}}(V, \mathcal{E})$ to be the index of the hypercohomology of this complex, i.e., the alternating sum of K-dimensions of the hypercohomology groups.

Lemma 7.6. *We have*

$$\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = n\chi(U) - \sum_{z \in Z} \operatorname{Irr}_z(\nabla) = n(2 - 2g(X) - \operatorname{length}(Z)) - \sum_{z \in Z} \operatorname{Irr}_z(\nabla). \tag{7.6.1}$$

Proof. Let K_0 be a subfield of K which is finitely generated over $\mathbb{Q}$ to which $X, Z, \mathcal{E}, \nabla$ can be descended. Then choose an embedding $K_0 \in \mathbb{C}$ and let $X_{\mathbb{C}}$ be the base extension of the descent of X, again equipped with a meromorphic vector bundle $\mathcal{E}$ and connection ∇. Note that $\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E})$ is computed by a spectral sequence in which one first computes the coherent cohomology of $\mathcal{E}$ and $\mathcal{E} \otimes \Omega$ separately. By the GAGA principle both over $\mathbb{C}$ [22, Exposé XII] and K [14, Example 3.2.6], these coherent cohomology groups can be computed equally well over any of X^{an}, X (or its descent to K_0), $X_{\mathbb{C}}$, or $X_{\mathbb{C}}^{\mathrm{an}}$. Consequently, despite the fact that the connection is only K-linear rather than $\mathcal{O}$-linear, we may nonetheless conclude that $\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = \chi_{\mathrm{dR}}(X_{\mathbb{C}}^{\mathrm{an}}, \mathcal{E})$. (As an aside, this argument recovers a comparison theorem of Baldassarri [2].)

To compute $\chi_{\mathrm{dR}}(X_{\mathbb{C}}^{\mathrm{an}}, \mathcal{E})$, we form a finite open covering $\{V_i\}_{i \in I}$ of $X_{\mathbb{C}}$ with the following properties.

- For each $i \in I$, V_i is isomorphic to a simply connected domain in $\mathbb{C}$, and $V_i \cap Z$ contains at most one point.
- For $i, j \in I$ distinct, $V_i \cap V_j$ is either empty or isomorphic to a simply connected domain in $\mathbb{C}$, and $V_i \cap V_j \cap Z = \emptyset$.
- For $i, j, k \in I$ pairwise distinct, $V_i \cap V_j \cap V_k = \emptyset$.

We then have

$$\chi_{\mathrm{dR}}(X_{\mathbb{C}}^{\mathrm{an}}, \mathcal{E}) = \sum_{i \in I} \chi_{\mathrm{dR}}(V_i, \mathcal{E}) - \sum_{i \neq j \in I} \chi_{\mathrm{dR}}(V_i \cap V_j, \mathcal{E}).$$

It then suffices to check that for $i, j \in I$ not necessarily distinct,

$$\chi_{\mathrm{dR}}(V_i \cap V_j, \mathcal{E}) = \begin{cases} -\operatorname{Irr}_z(\nabla) & V_i \cap V_j \cap Z = \{z\} \\ 1 & V_i \cap V_j \cap Z = \emptyset. \end{cases} \tag{7.6.2}$$

In case $V_i \cap V_j \cap Z = \emptyset$, this is immediate because the restriction of $\mathcal{E}$ to $V_i \cap V_j$ is trivial. In case $V_i \cap V_j \cap Z = \{z\}$, we may similarly replace $V_i \cap V_j$ with a small open disc around z, and then invoke the Deligne-Malgrange interpretation of irregularity as the local index of meromorphic de Rham cohomology on a punctured disc [30, Théorème 3.3(d)]. $\square$

Theorem 7.7 (Global index formula)**.** *We have*

$$\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = n\chi(U) - \int \Delta h(\mathcal{N}) = n(2 - 2g(X) - \mathrm{length}(Z)) - \int \Delta h(\mathcal{N}). \tag{7.7.1}$$

Proof. This follows by comparing Lemma 7.4 with Lemma 7.6. □

Remark 7.8. It may not be immediately obvious why Theorem 7.7 is of value, i.e., why it is useful to express the index of de Rham cohomology in terms of convergence polygons instead of irregularity. It turns out that there is a profound difference between the behavior of the index in the complex-analytic and nonarchimedean settings. In the complex case, for any open analytic subspace V of $X^{\mathrm{an}}_{\mathbb{C}}$, we have

$$\chi_{\mathrm{dR}}(V, \mathcal{E}) = n\chi(V \cap U^{\mathrm{an}}) - \sum_{z \in V \cap Z} \mathrm{Irr}_z(\nabla) \tag{7.8.1}$$

by the same argument as in the proof of (7.6.2). In particular, $\chi_{\mathrm{dR}}(V, \mathcal{E}) \neq \chi_{\mathrm{dR}}(V - Z, \mathcal{E})$; that is, the index of de Rham hypercohomology depends on whether we allow poles or essential singularities at the points of Z. By contrast, in the nonarchimedean case, these two indices coincide under a suitable technical hypothesis to ensure that they are both defined; see Example 7.9 for a simple example and Corollary 7.13 for the general case (and Example 7.19 and Remark 7.20 for a counterexample failing the technical hypothesis). This means that in the nonarchimedean case, the "source" of the index of de Rham cohomology is not irregularity, but rather the Laplacian of the convergence polygon (see Theorem 7.12).

Example 7.9. Consider the connection associated to Example 4.2 as per Remark 5.7; note that $\mathrm{Irr}_0(\nabla) = 0$, $\mathrm{Irr}_\infty(\nabla) = 1$, so $\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = -1$. For $\alpha > 0$, let V_α, W_α be the subspace $|z| < \alpha$, $|z| > \alpha$ of X^{an}. By Mayer-Vietoris,

$$\chi_{\mathrm{dR}}(V_\beta, \mathcal{E}) + \chi_{\mathrm{dR}}(W_\alpha, \mathcal{E}) - \chi_{\mathrm{dR}}(V_\beta \cap W_\alpha, \mathcal{E}) = \chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = -1. \tag{7.9.1}$$

On the other hand, $\chi_{\mathrm{dR}}(V_\alpha, \mathcal{E})$ (resp. $\chi_{\mathrm{dR}}(W_\alpha, \mathcal{E})$) equals the index of the operator $y \mapsto y' - y$ on Laurent series in z convergent for $|z| < \alpha$ (resp. in z^{-1} convergent for $|z^{-1}| < \alpha^{-1}$). If $f = \sum_n f_n z^{-n}$, $g = \sum_n g_n z^{-n}$ are two such series, then the equation $g = f' - f$ is equivalent to

$$f_n = -g_n - (n-1) f_{n-1} \qquad (n \in \mathbb{Z}). \tag{7.9.2}$$

Suppose first that $p^{-1/(p-1)} < \alpha < \beta$. Then given $g \in K((z^{-1}))$, we may solve uniquely for $f \in K((z^{-1}))$, and if g converges on $W_\alpha - \{\infty\}$, then so does f. We thus compute that

$$\chi_{\mathrm{dR}}(V_\beta, \mathcal{E}) = -1, \quad \chi_{\mathrm{dR}}(W_\alpha, \mathcal{E}) = 0, \quad \chi_{\mathrm{dR}}(V_\beta \cap W_\alpha, \mathcal{E}) = 0;$$

namely, the second equality is what we just computed, the third follows from Robba's index formula [42] (see also [28, Lemma 3.7.5]), and the first follows from the other two plus (7.9.1). In particular, the local index at ∞ equals 0, whereas in the complex-analytic setting it equals -1 by the Deligne-Malgrange formula (see the proof of Lemma 7.6).

Suppose next that $\alpha < \beta < p^{-1/(p-1)}$. Then by contrast, we have

$$\chi_{\mathrm{dR}}(V_\beta, \mathcal{E}) = 0, \quad \chi_{\mathrm{dR}}(W_\alpha, \mathcal{E}) = -1, \quad \chi_{\mathrm{dR}}(V_\beta \cap W_\alpha, \mathcal{E}) = 0;$$

namely, the first and third equalities follow from the triviality of ∇ on V_β, and the second follows from the other two plus (7.9.1).

With Example 7.9 in mind, we now describe a local refinement of Theorem 7.7, in which we dissect the combinatorial formula for the index into local contributions.

Definition 7.10. For $x \in U^{\mathrm{an}} \cap \Gamma_{X,Z}$, let $\mathrm{val}_\Gamma(x)$ be the valence of x as a vertex of $\Gamma_{X,Z}$, taking $\mathrm{val}_\Gamma(x) = 2$ when x lies on the interior of an edge. (We refer to *valence* instead of *degree* to avoid confusion with degrees of morphisms.) For $x \in U^{\mathrm{an}}$, define

$$\chi_x(\mathcal{E}) = \begin{cases} n(2 - 2g(C_x) - \mathrm{val}_\Gamma(x)) - (\Delta h(\mathcal{N}))_x & x \in \Gamma_{X,Z} \\ -\Delta h(\mathcal{N})_x & x \notin \Gamma_{X,Z}. \end{cases} \tag{7.10.1}$$

Let $\chi(\mathcal{E})$ be the $\mathbb{R}$-valued measure whose value on a continuous function $f : U^{\mathrm{an}} \to \mathbb{R}$ is $\sum_{x \in U^{\mathrm{an}}} f(x)\chi_x(\mathcal{E})$.

Lemma 7.11. *We have*

$$\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = n\chi(U) - \int \Delta h(\mathcal{N}) = \int \chi(\mathcal{E}).$$

Proof. This follows from Theorem 7.7 plus the identity

$$\sum_{x \in U^{\mathrm{an}} \cap \Gamma_{X,Z}} (2 - 2g(C_x) - \mathrm{val}_\Gamma(x)) = 2 - 2g(X) - \mathrm{length}(Z),$$

which amounts to the combinatorial formula for the genus of an analytic curve [6, §4.16]. □

Theorem 7.12 (Local index formula)**.** *Let V be an open subspace of X^{an} which is the retraction of an open subspace of $\Gamma_{X,Z}$. If $p > 0$, assume some additional technical hypotheses (see Remark 7.14). Then*

$$\chi_{\mathrm{dR}}(V, \mathcal{E}) = \int_{V \cap U^{\mathrm{an}}} \chi(\mathcal{E}).$$

Proof. See [34, Theorem 3.5.2]. □

Corollary 7.13. *With hypotheses as in Theorem 7.12, $\chi_{\mathrm{dR}}(V, \mathcal{E}) = \chi_{\mathrm{dR}}(V \cap U^{\mathrm{an}}, \mathcal{E})$; that is, the index of de Rham hypercohomology is the same whether we allow poles or essential singularities at Z.*

Remark 7.14. Let Γ be a strict skeleton for which the conclusion of Theorem 5.5 holds. For v a vertex of Γ, define the *star* of v, denoted $\star_v$, as the union of v and the interiors of the edges of Γ incident to v. Let $\pi_\Gamma : X \to \Gamma$ be the retraction onto Γ, through which $\mathcal{N}$ factors. Under the hypotheses of Theorem 7.12, we have

$$\chi_{\mathrm{dR}}(\pi_\Gamma^{-1}(\star_v), \mathcal{E}) = \chi_v(\mathcal{E}), \qquad \chi_{\mathrm{dR}}(\pi_\Gamma^{-1}(\star_v \cap \star_w), \mathcal{E}) = 0. \tag{7.14.1}$$

We can then recover Theorem 7.7 from (7.14.1) by using Mayer-Vietoris (and GAGA over K; see Remark 7.8) to write

$$\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = \sum_v \chi_{\mathrm{dR}}(\pi_\Gamma^{-1}(\star_v), \mathcal{E}) - \sum_{v \neq w} \chi_{\mathrm{dR}}(\pi_\Gamma^{-1}(\star_v \cap \star_w), \mathcal{E});$$

one may similarly deduce Theorem 7.12 from (7.14.1).

Remark 7.15. For a given connection, one can extend the range of applicability of Theorem 7.12 by enlarging the set Z. Let us examine closely what happens if we add one additional point $z' \in U(K)$ to Z. On one hand, by Lemma 7.6, enlarging Z has the effect of decreasing $\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E})$ by n.

On the other hand, let $V_{z'}$ be the maximal open disc in U^{an} containing z'; its generic point is some $y' \in U^{\mathrm{an}} \cap \Gamma_{X,Z}$. If we put $Z' = Z \cup \{z'\}$, then $\Gamma_{X,Z'}$ is the union of $\Gamma_{X,Z}$ (subdivided at x' if necessary) with the path from x' to z' in $V_{z'}$. It follows that enlarging Z does not change $\chi_x(\mathcal{E})$ for any $x \in U^{\mathrm{an}} - V_{z'} - \{y\}$. It follows that $\sum_{x \in V_{z'} \cup \{y\}} \chi_x(\mathcal{E})$ must decrease by n, but it is difficult to predict in advance for which x the change occurs.

Remark 7.16. One of the main reasons we have restricted attention to meromorphic connections on proper curves is that in this setting, Theorem 5.5 ensures that $\chi_x(\mathcal{E}) = 0$ for all but finitely many $x \in U^{\mathrm{an}}$. It is ultimately more natural to state Theorem 7.12 for connections on open analytic curves, as is done in [34, Theorem 3.8.10]; however, this requires some additional hypotheses to ensure that $\chi(\mathcal{E})$ is a finite measure.

Definition 7.17. Assume $p > 0$. A *p-adic Liouville number* is an element $x \in \mathbb{Z}_p - \mathbb{Z}$ such that

$$\liminf_{m \to \infty} \frac{\{|y| : y \in \mathbb{Z}, y - x \in p^m \mathbb{Z}_p\}}{m} < +\infty.$$

As in the classical case, p-adic Liouville numbers are always transcendental [19, Proposition VI.1.1].

Remark 7.18. Assume $p > 0$. The technical hypotheses of Theorem 7.12 are needed to guarantee the existence of the indices appearing in (7.14.1). In case ∇ has a regular singularity at $z \in Z$ with all exponents in $\mathbb{Z}_p$, these hypotheses include the condition that no two exponents of ∇ at z differ by a p-adic Liouville number; see Example 7.19 for a demonstration of the necessity of such a condition.

Unfortunately, the full hypotheses are somewhat more complicated to state. They arise from the fact that with notation as in Remark 7.14, one can separate off a maximal component of $\mathcal{E}$ on $\pi_\Gamma^{-1}(\star_v \cap \star_w)$ which satisfies the Robba condition, to which one may associate some p-adic numbers playing the role of exponents; the hypothesis is that (for any particular v, w) no two of these numbers differ by a p-adic Liouville numbers. The difficulty is that the definition of these *p-adic exponents*, due to Christol and Mebkhout (and later simplified by Dwork) is somewhat indirect; they occur as "resonant frequencies" for a certain action by the group of p-power roots of unity, which are hard to control except in some isolated cases where they are forced to be rational numbers (e.g., Picard-Fuchs equations, a/k/a Gauss-Manin connections, or connections arising from F-isocrystals in the theory of crystalline cohomology). See [25, Chapter 13] for more discussion.

Example 7.19. Assume $p > 0$ and take $X = \mathbb{P}^1_K$, $Z = \{0, \infty\}$. Take $\mathcal{E}$ to be free of rank 1 with the action of ∇ given by

$$\nabla(f) = \lambda f \frac{dz}{z} + df$$

for some $\lambda \in K$. For α, β with $0 < \alpha < \beta$, let V be the open annulus $\alpha < |z| < \beta$. The 1-forms on V are series $\sum_{n=-\infty}^{\infty} c_n z^n \frac{dz}{z}$ such that $|c_n|\alpha^n \to 0$ as $n \to -\infty$ and $|c_n|\beta^n \to 0$ as $n \to +\infty$.

If $\lambda = 0$, then $|c_n|\alpha^n \to 0$ if and only if $|c_n/n|\alpha^n \to 0$ and similarly for β, so every 1-form on V with $c_0 = 0$ is in the image of ∇. It follows that if $\lambda \in \mathbb{Z}$, then the kernel and cokernel of ∇ on V are both 1-dimensional, so $\chi_{\mathrm{dR}}(U^{\mathrm{an}}, \mathcal{E}) = 0$.

If $\lambda \in K - \mathbb{Z}$, then a 1-form is in the image of ∇ on V if and only if $|c_n/(n-\lambda)|\alpha^n \to 0$ as $n \to -\infty$ and $|c_n/(n-\lambda)|\beta^n \to 0$ as $n \to +\infty$. This always holds if λ is not a p-adic Liouville number (see [19, §VI.1] or [25, Proposition 13.1.4]); otherwise, one shows that ∇ has infinite-dimensional cokernel on U^{an}, so $\chi_{\mathrm{dR}}(U^{\mathrm{an}}, \mathcal{E})$ is undefined.

Remark 7.20. Example 7.19 provides an example showing that the equality $\chi_{\mathrm{dR}}(V, \mathcal{E}) = \chi_{\mathrm{dR}}(V \cap U^{\mathrm{an}}, \mathcal{E})$ of Corollary 7.13 cannot hold without conditions on p-adic Liouville numbers: in this example, for all λ, $\chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) = 0$ by Lemma 7.6; but when λ is a p-adic Liouville number, $\chi_{\mathrm{dR}}(U^{\mathrm{an}}, \mathcal{E})$ is undefined.

Remark 7.21. The net result of Remark 7.18 is that in general, one can only view $\chi_x(\mathcal{E})$ as a *virtual local index* of $\mathcal{E}$ at x, not a true local index. Nonetheless, this interpretation can be used to predict combinatorial properties of the convergence polygon which often continue to hold even without restrictions on p-adic exponents. For example, Theorem 6.5 corresponds to the fact that if V is an open disc in U^{an}, then the dimension of the cokernel of ∇ on V is nonnegative; this argument appears in the proof of Theorem 6.5 in the case $p = 0$ (see Theorem B.4).

Remark 7.22. In light of Remark 7.21, one might hope to establish some inequalities on $\chi_x(\mathcal{E})$. One might first hope to refine Theorem 7.23 by analogy with (7.8.1), by proving that the measure $\Delta h(\mathcal{N})$ is nonnegative; however, this fails already in simple examples such as Example 7.25.

On the other hand, since our running hypothesis is that $\chi(U) \leq 0$, Theorem 7.7 and Lemma 7.11 imply that

$$\sum_{x \in U^{\mathrm{an}}} \chi_x(\mathcal{E}) = \chi_{\mathrm{dR}}(X^{\mathrm{an}}, \mathcal{E}) \leq n\chi(U) \leq 0.$$

One might thus hope to refine Theorem 7.23 by proving that $\chi_x(\mathcal{E}) \leq 0$ for all $x \in U^{\mathrm{an}}$. Unfortunately, this is not known (and may not even be safe to conjecture) in full generality, but see Theorem 7.23 for some important special cases.

Theorem 7.23. *Choose $x \in U^{\mathrm{an}}$.*

(a) *If $\mathcal{N}(x)$ has only nonzero slopes, then $\chi_x(\mathcal{E}) = 0$.*
(b) *If $x \in \Gamma_{X,Z}$, then $\chi_x(\mathcal{E}) \leq 0$.*
(c) *If $p = 0$, then $\chi_x(\mathcal{E}) \leq 0$.*

Proof. For (a), see [33] or [28] or [5]; a similar argument implies (b) because in this case the zero slopes are forced to make a nonpositive contribution to the index. For (c), see Theorem B.7. □

Remark 7.24. The proof of Theorem 7.23 can also be used to quantify the extent to which $\mathcal{N}$ fails to factor through the retract onto $\Gamma_{X,Z}$, and hence to help identify a suitable skeleton Γ for which the conclusion of Theorem 5.5 holds. To be precise, for $x \in \Gamma_{X,Z}$, if the restriction of $\mathcal{N}$ to $\Gamma_{X,Z}$ is harmonic at x, then $\mathcal{N}$ is constant on the fiber at x of the retraction of X^{an} onto $\Gamma_{X,Z}$.

Example 7.25. Let h be a nonzero rational function on X, and take Z to be the pole locus of h. Let $\mathcal{E}$ be the free bundle on a single generator $\mathbf{v}$ equipped with the connection

$$\nabla(f\mathbf{v}) = \mathbf{v} \otimes (df + f\,dh).$$

For each $z \in Z$, $\mathrm{Irr}_z(\nabla)$ equals the multiplicity of z as a pole of Z. By Lemma 7.11,

$$\sum_{x \in U^{\mathrm{an}}} \chi_x(\mathcal{E}) = \chi(U) - m$$

where m is the number of poles of h counted with multiplicity.

Suppose now that there exists a point $x \in U^{\mathrm{an}}$ with $g(C_x) > 0$. By multiplying h by a suitably large element of K, we can ensure that $\mathcal{N}(x)$ has positive slope. In this case, by Theorem 7.23 we must have

$$\chi_x(\mathcal{E}) = 0 > 2 - 2g(C_x) - \mathrm{val}_\Gamma(x).$$

In particular, while $\int \Delta h(\mathcal{N}) = \sum_{z \in Z} \mathrm{Irr}_z(\nabla) \geq 0$, the measure $\Delta h(\mathcal{N})$ is not necessarily nonnegative.

8. Ramification of finite morphisms

Hypothesis 8.1. Throughout §8, let $f : Y \to X$ be a finite flat morphism of curves over K of degree d whose restriction to U is étale and Galois with Galois group G. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful representation of G on an n-dimensional vector space V over K.

Definition 8.2. Equip $\mathcal{E}_V = \mathcal{O}_Y \otimes_K V^\vee$ with the diagonal action of G induced by the Galois action on $\mathcal{O}_Y$ and the action via $\rho^\vee$ on $V^\vee$. By faithfully flat descent, the restriction of $\mathcal{E}_V$ to $f^{-1}(U)$ descends uniquely to a vector bundle $\mathcal{E}$ on U; moreover, the trivial connection on $\mathcal{E}_V$ induces a connection ∇ on $\mathcal{E}$. We are thus in the situation of Hypothesis 5.1. Note that $\mathrm{Irr}_z(\nabla) = 0$ for all $z \in Z$.

Remark 8.3. One way to arrive at Hypothesis 8.1 is to start with a non-Galois cover $g : W \to X$, let f be the Galois closure, and let ρ be the representation of G induced by the trivial representation of the subgroup of G fixing W. In this case, $\mathcal{E}$ is just the pushforward of the trivial connection on $g^{-1}(U)$.

Definition 8.4. For L a nonarchimedean field over K, let $\mathbb{C}_L$ denote a completed algebraic closure of K, and let B_L be the subset of X_L for which $x \in B_L$ if and only if the preimage of x in $Y^{\mathrm{an}}_{\mathbb{C}_L}$ does not consist of d distinct points. By definition, the construction of B_L commutes with base extension to a completed algebraic extension, but not more generally: B_L only contains points of types 2 or 3, whereas a sufficiently large base extension would add some points of type 1.

Theorem 8.5. *For L a nonarchimedean field over K and $x \in U(L)$, $e^{-s_1(\mathcal{N}(x))}$ equals the supremum of $\rho \in (0,1]$ such that $U_{x,\rho} \cap B_L = \emptyset$.*

Proof. We may assume without loss of generality that L is itself algebraically closed. If $U_{x,\rho} \cap B_L = \emptyset$, then the map $f^{-1}(U_{x,\rho}) \to U_{x,\rho}$ is a covering space map of topological spaces; since $U_{x,\rho}$ is contractible and hence simply connected, $f^{-1}(U_{x,\rho})$ splits topologically as a disjoint union of copies of $U_{x,\rho}$. From this it follows easily that the restriction of $\mathcal{E}$ to $U_{x,\rho}$ is trivial. Conversely, if this restriction is trivial, then the restriction of $\mathcal{E}_V$ to $f^{-1}(U_{x,\rho})$ admits a family of idempotent elements corresponding to the splitting of $f^{-1}(x)$ into d singleton sets; these then define a splitting of $f^{-1}(U_{x,\rho})$ into d disjoint sets, proving that $U_{x,\rho} \cap B_L = \emptyset$. $\square$

Remark 8.6. One may view Theorem 8.5 as saying that under a suitable normalization, $e^{-s_1(\mathcal{N}(x))}$ measures the distance from x to B_L. This suggests the interpretation of B_L as an "extended ramification locus" of the map f; for maps from $\mathbb{P}^1_K$ to itself, this interpretation has been adopted in the context of nonarchimedean dynamics (e.g., see [20, 21]). However, this picture is complicated by the fact that B_L is not stable under base extension: even for $x \in B_L$, the "distance from x to itself" must be interpreted as a nonzero quantity. In any case, Theorem 8.5 suggests the possibility of relating the full convergence polygon to more subtle measures of ramification, such as those considered recently by Temkin [13, 44]. One older result in this direction is the theorem of Matsuda; see Theorem 8.7 below.

Theorem 8.7. *Assume that K has perfect residue field of characteristic $p > 0$. Suppose that $x \in \Gamma$ is of type 2, choose $y \in f^{-1}(\{x\})$, and suppose that $\mathcal{H}(y)$ is unramified over $\mathcal{H}(x)$.*

(a) *The polygon $\mathcal{N}(x)$ is zero.*
(b) *Let $\vec{t}$ be a branch of X at x, and let v be the point on C_x corresponding to $\vec{t}$ (see Definition 6.1). Let $\overline{\rho} : \mathrm{Gal}(\kappa_{\mathcal{H}(y)}/\kappa_{\mathcal{H}(x)}) \to V$ be the representation induced by ρ. Then $\partial_{\vec{t}}(\mathcal{N})$ computes the Newton polygon associated to the (wild) ramification filtration of $\overline{\rho}$ at v. In particular, $\partial_{\vec{t}}(h(\mathcal{N}))$ computes the Swan conductor of $\overline{\rho}$ at v.*

Proof. See [31]. $\square$

Example 8.8. Take $K = \mathbb{Q}_p(\zeta_p)$, $X = \mathbb{P}^1_K$, $Z = \{0, \infty\}$, $Y = \mathbb{P}^1_K$, let $f : Y \to X$ be the map $z \mapsto z^p$, identify G with $\mathbb{Z}/p\mathbb{Z}$ so that $1 \in \mathbb{Z}/p\mathbb{Z}$ corresponds to the map $z \mapsto \zeta z$ on $f^{-1}(U)$, and let $\rho : G \to \mathrm{GL}_1(K)$ be the character taking 1 to ζ_p^{-1}. Then $\mathcal{E}$ is free on a single generator $\mathbf{v}$ satisfying $\nabla(\mathbf{v}) = -\frac{1}{p} z^{-1}\mathbf{v}$. This is also the connection obtained from Example 4.3 by applying Remark 5.7.

Put $\omega = p^{-1/(p-1)} = |\zeta_p - 1|$. For each $x \in U$, we may define the *normalized diameter* of x as an element of $[0, 1]$ defined as follows: choose an extension L of K such that x lifts to some $\tilde{x} \in U(L)$, then take the infimum of all $\rho \in (0, 1]$ such that $U_{\tilde{x},\rho}$ meets the inverse image of x in U_L^{an} (or 1 if no such ρ exists). With this definition, for any L, the set B_L consists of Z plus all points with normalized diameter in $[\omega^p, 1]$.

Now let x be the generic point of the disc $|z - 1| \leq \omega^p$, which we can also write as $|u| \leq 1$ for $u = (z - 1)/(\zeta_p - 1)^p$. Choose Γ to consist of the path from x to x_1 together with the paths from x_1 to 0 and ∞. Let y be the unique preimage of x in Y. Let t be the coordinate $(z - 1)/(\zeta_p - 1)$ on Y; then $\kappa_{\mathcal{H}(x)} = \mathbb{F}_p(\overline{u})$ while $\kappa_{\mathcal{H}(y)} = \mathbb{F}_p(\overline{t})$ where $\overline{t}^p - \overline{t} = \overline{u}$. For $\vec{t}$ the branch of x towards x_1, we have $\partial_{\vec{t}}(h(\mathcal{N})) = 1$; as predicted by Theorem 8.7, this equals the Swan conductor of the residual extension at ∞.

Remark 8.9. To generalize Theorem 8.7 to cases where $\mathcal{H}(y)$ is ramified over $\mathcal{H}(x)$, it may be most convenient to use Huber's ramification theory for adic curves [23], possibly as refined in [13, 44]. In a similar vein, the global index formula (Theorem 7.7) is essentially the Riemann-Hurwitz formula for the map f, in which case it should be possible to match up the local contributions appearing in Theorem 7.7 with ramification-theoretic local contributions.

Remark 8.10. Suppose that $p > 0$, $X = \mathbb{P}^1_K$, and f extends to a finite flat morphism of smooth curves over $\mathfrak{o}_K$ with target $\mathbb{P}^1_{\mathfrak{o}_K}$. Then $\mathcal{E}$ admits a unit-root Frobenius structure in a neighborhood of x_1, from which it follows that $\mathcal{E}$ satisfies the Robba condition at x_1. If in addition the points of Z have distinct projections to $\mathbb{P}^1_k$, then by Remark 5.11, $\mathcal{E}$ satisfies the Robba condition everywhere. By Remark 6.4, this implies that $\mathcal{E}$ has exponents in $\mathbb{Z}_p$, and hence f is tamely ramified (i.e., the stabilizer of each point of Y has order coprime to p). By contrast, if the points of Z do not have distinct projections to $\mathbb{P}^1_k$, then f need not be tamely ramified; see §11.

9. Artin–Hasse exponentials and Witt vectors

We will conclude by specializing the previous discussion to cyclic covers of discs in connection with the Oort local lifting problem. In preparation for this, we need to recall some standard constructions of p-adic analysis.

Hypothesis 9.1. Throughout §9, fix a prime p and a positive integer n.

Definition 9.2. The *Artin-Hasse exponential series* at p is the formal power series

$$E_p(t) := \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right).$$

Lemma 9.3. *We have $E_p(t) \in \mathbb{Z}_{(p)}[\![t]\!]$. In particular, $E_p(t)$ converges for $|t| < 1$.*

Proof. See for instance [25, Proposition 9.9.2]. $\square$

Lemma 9.4. *Let $\mathbb{Z}_{(p)}\langle t\rangle$ be the subring of $\mathbb{Z}_{(p)}[\![t]\!]$ consisting of series $\sum_{i=0}^{\infty} c_i t^i$ for which the c_i converge p-adically to 0 as $i \to \infty$. Then if we define the power series*

$$f(z,t) := \frac{E_p(zt)E_p(t^p)}{E_p(t)E_p(zt^p)} \in \mathbb{Z}_{(p)}[\![t, 1-z]\!]$$

using Lemma 9.3, we have

$$f(z,t) \in z + (t-1)\mathbb{Z}_{(p)}\langle t\rangle[\![1-z]\!].$$

Proof. Write $f(z,t) = \exp g(z,t)$ with

$$\begin{aligned}
g(z,t) &= \sum_{i=0}^{\infty} \frac{(z^{p^i}-1)(t^{p^i} - t^{p^{i+1}})}{p^i} \\
&= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} (-1)^j \binom{p^i}{j} (1-z)^j \frac{t^{p^i} - t^{p^{i+1}}}{p^i} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} (1-z)^j \sum_{i=0}^{\infty} \binom{p^i-1}{j-1} (t^{p^i} - t^{p^{i+1}}) \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} (1-z)^j \left(\binom{0}{j-1} t + \sum_{i=1}^{\infty} \left(\binom{p^i-1}{j-1} - \binom{p^{i-1}-1}{j-1} \right) t^{p^i} \right).
\end{aligned}$$

For each fixed j, $\binom{p^i-1}{j-1}$ converges p-adically to $\binom{-1}{j-1} = (-1)^{j-1}$ as $i \to \infty$. It follows that $g(z,t) \in \mathbb{Q}_{(p)}\langle t\rangle[\![1-z]\!]$ and

$$g(z,t) \equiv -\sum_{j=1}^{\infty} \frac{(1-z)^j}{j} \pmod{(t-1)\mathbb{Q}_{(p)}\langle t\rangle[\![1-z]\!]}.$$

This then implies that $f(z,t) \in \mathbb{Q}_{(p)}\langle t\rangle[\![1-z]\!]$ and $f(z,t) \equiv z \pmod{(1-t)\mathbb{Q}_{(p)}\langle t\rangle[\![1-z]\!]}$. Since $f(z,t)$ also belongs to $\mathbb{Z}_{(p)}[\![t, 1-z]\!]$ by Lemma 9.3, we may deduce the claimed inclusion. $\square$

Definition 9.5. Let W_n denote the p-typical Witt vector functor. Given a ring R, the set $W_n(R)$ consists of n-tuples $\underline{a} = (a_0, \dots, a_{n-1})$, and the arithmetic operations on $W_n(R)$ are characterized by functoriality in R and the property that the *ghost map* $w : W_n(R) \to R^n$ given by

$$(a_0, \dots, a_{n-1}) \mapsto (w_0, \dots, w_{n-1}), \qquad w_i = \sum_{j=0}^{i} p^j a_j^{p^{i-j}} \tag{9.5.1}$$

is a ring homomorphism for the product ring structure on R^n. For any ideal I of R, let $W_n(I)$ denote the subset of $W_n(R)$ consisting of n-tuples with components in I; since $W_n(I) = \ker(W_n(R) \to W_n(R/I))$, it is an ideal of $W_n(R)$.

Various standard properties of Witt vectors may be derived by using functoriality to reduce to polynomial identities over $\mathbb{Z}$, then checking these over $\mathbb{Q}$ using the fact that the ghost map is a bijection if $p^{-1} \in R$. Here are two key examples.

(i) Define the *Teichmüller map* sending $r \in R$ to $[r] := (r, 0, \dots, 0) \in W_n(R)$. Then this map is multiplicative: for all $r, s \in R$, $[rs] = [r][s]$.

(ii) Define the *Verscheibung map* sending $\underline{a} \in W_n(R)$ to $V(\underline{a}) := (0, a_0, \dots, a_{n-2}) \in W_n(R)$. Then this map is additive: for all $\underline{a}, \underline{b} \in W_n(R)$, $V_n(\underline{a} + \underline{b}) = V_n(\underline{a}) + V_n(\underline{b})$.

Definition 9.6. In case R is an $\mathbb{F}_p$-algebra, the Frobenius endomorphism $\varphi : R \to R$ extends by functoriality to $W_n(R)$ and satisfies

$$p\underline{a} = (V \circ \varphi)(\underline{a}) \qquad (\underline{a} \in R);$$

see for instance [24, § 0.1]. It follows that for general R, if we define the map σ sending $\underline{a} \in W_n(R)$ to $\sigma(\underline{a}) = (a_0^p, \dots, a_{n-1}^p) \in W_n(R)$, then

$$p\underline{a} - (V \circ \sigma)(\underline{a}) \in W_n(pR) \qquad (\underline{a} \in W_n(R)). \tag{9.6.1}$$

Beware that σ is in general not a ring homomorphism.

Definition 9.7. By Lemma 9.3, we have

$$E_{n,p}(t) := \frac{E_p(\zeta_{p^n} t)}{E_p(t)} = \exp\left(\sum_{i=0}^{n-1} (\zeta_{p^{n-i}} - 1)\frac{t^{p^i}}{p^i}\right) \in \mathbb{Z}_{(p)}[\zeta_{p^n}][\![t]\!].$$

We may also define

$$E_{n,p}(\underline{a}) := \prod_{i=0}^{n-1} E_{n-i,p}(a_i) \in \mathbb{Z}_{(p)}[\zeta_{p^n}][\![a_0, \dots, a_{n-1}]\!],$$

which we may also write as

$$E_{n,p}(\underline{a}) = \exp\left(\sum_{i=0}^{n-1} (\zeta_{p^{n-i}} - 1)\frac{w_i}{p^i}\right) \tag{9.7.1}$$

for w_i as in (9.7.1). Consequently, in $\mathbb{Z}_{(p)}[\zeta_{p^n}][\![a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}]\!]$,

$$E_{n,p}(\underline{a})E_{n,p}(\underline{b}) = E_{n,p}(\underline{a} + \underline{b}). \tag{9.7.2}$$

Definition 9.8. By Lemma 9.3, we may define the formal power series

$$F_{n,p}(t) := \frac{E_{n,p}(t)}{E_{n,p}(t^p)} = \exp\left(\sum_{i=0}^{n-1} \frac{(\zeta_{p^{n-i}} - 1)(t^{p^i} - t^{p^{i+1}})}{p^i}\right) \in \mathbb{Z}_{(p)}[\zeta_{p^n}][\![t]\!]$$

and

$$F_{n,p}(\underline{a}) := \frac{E_{n,p}(\underline{a})}{E_{n,p}(\sigma(\underline{a}))} = \prod_{i=0}^{n-1} F_{n-i,p}(a_i) \in \mathbb{Z}_{(p)}[\zeta_{p^n}][\![a_0, \dots, a_{n-1}]\!].$$

By (9.7.1), in $\mathbb{Z}_{(p)}[\zeta_{p^n}][\![a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}]\!]$ we have

$$F_{n,p}(\underline{a})F_{n,p}(\underline{b}) = F_{n,p}(\underline{a} + \underline{b})E_{n,p}(\sigma(\underline{a} + \underline{b}) + \sigma(\underline{a}) + \sigma(\underline{b})). \tag{9.8.1}$$

We will see shortly (Lemma 9.10) that $F_{n,p}(t)$ has radius of convergence greater than 1, which implies an analogous assertion for $F_{n,p}(\underline{a})$. For $p > 2$, this is shown in [31, Proposition 1.10] using a detailed computational argument; our argument follows the more conceptual approach given in [36, Theorem 2.5].

Definition 9.9. By Lemma 9.3, we may define the formal power series

$$G_{n,p}(\underline{a}) = \frac{E_{n,p}(p\underline{a})}{E_{n-1,p}(\underline{a})} \in \mathbb{Z}_{(p)}[\zeta_{p^n}][\![a_0, \dots, a_{n-1}]\!].$$

By (9.7.2),

$$G_{n,p}(\underline{a})G_{n,p}(\underline{b}) = G_{n,p}(\underline{a} + \underline{b}). \tag{9.9.1}$$

We also have

$$G_{n,p}(\underline{a}) = E_{n,p}(p\underline{a} - (0, a_0^p, \dots, a_{n-2}^p))F_{n-1,p}(\underline{a})^{-1}. \tag{9.9.2}$$

Lemma 9.10. (a) *We have*

$$G_{n,p}(\underline{a}) \in \mathbb{Z}_{(p)}[\zeta_{p^n}][\![(\zeta_{p^n} - 1)a_0, \dots, (\zeta_p - 1)a_{n-1}]\!].$$

(b) *The power series $F_{n,p}(\underline{a})$ converges on a polydisc with radius of convergence strictly greater than* 1.

Proof. Write

$$\begin{aligned} G_{n,p}(\underline{a}) &= \prod_{j=0}^{n-1} \exp\left(\sum_{i=0}^{n-j-1} (p(\zeta_{p^{n-j-i}} - 1) - (\zeta_{p^{n-j-1-i}} - 1))\frac{a_j^{p^i}}{p^i}\right) \\ &= \prod_{j=0}^{n-1} \exp\left(\sum_{i=0}^{n-j-1} (\zeta_{p^{n-j-i}} - 1)(p - 1 - \zeta_{p^{n-j}} - \cdots - \zeta_{p^{n-j}}^{p-1})\frac{a_j^{p^i}}{p^i}\right) \\ &= \prod_{j=0}^{n-1} E_{n,p}\left((p - 1 - [\zeta_{p^{n-j}}] - \cdots - [\zeta_{p^{n-j}}^{p-1}])[a_j]\right). \end{aligned}$$

Note that $p-1-[\zeta_{p^{n-j}}]-\cdots-[\zeta_{p^{n-j}}^{p-1}]$ maps to zero in $W_{n-j}(\mathbb{F}_p)$ and so belongs to $W_{n-j}((\zeta_{p^{n-j}} - 1)\mathbb{Z}_{(p)}[\zeta_{p^{n-j}}])$. This proves (a). To prove (b), combine (a) with (9.9.2) and (9.6.1). ☐

Lemma 9.11. *For $m \in \mathbb{Z}$, in $\mathbb{Z}_p[\zeta_{p^n}][\![a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}]\!]$ we have*

$$F_{n,p}(\underline{a} + \underline{b} + m)F_{n,p}(\underline{a})^{-1}E_{n,p}(\sigma(\underline{a} + \underline{b} + m) - \sigma(\underline{a}) - \underline{b} - m) = \zeta_{p^n}^m. \tag{9.11.1}$$

Note that the presence of m prevents us from heedlessly applying (9.7.2), because for instance $E_{n,p}(m)$ does not make sense. (We like to think of this as an example of *conditional convergence* in a nonarchimedean setting.) Similarly, we must work over $\mathbb{Z}_p$ rather than $\mathbb{Z}_{(p)}$.

Proof. Convergence of $F_{n,p}(\underline{a} + \underline{b} + m)$ is guaranteed by Lemma 9.10. Convergence of $E_{n,p}(\sigma(\underline{a} + \underline{b} + m) - \sigma(\underline{a}) - \underline{b} - m)$ is guaranteed by Lemma 9.3 and the fact that

$$\sigma(\underline{a} + \underline{b} + m) - \sigma(\underline{a}) - \underline{b} - m \in W((\zeta_{p^n} - 1)\mathbb{Z}_p[\zeta_{p^n}][\![a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}]\!]).$$

Thus the left side of (9.11.1) is well-defined. Using (9.7.2), we see that this quantity is constant as a power series in $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}$; it thus remains to prove that

$$F_{n,p}(m)E_{n,p}(\sigma(m) - m) = \zeta_{p^n}^m. \tag{9.11.2}$$

Using (9.7.2) and (9.8.1), we see that for $m, m' \in \mathbb{Z}$,

$$\begin{aligned} &F_{n,p}(m)E_{n,p}(\sigma(m) - m)F_{n,p}(m')E_{n,p}(\sigma(m') - m') \\ &\quad = F_{n,p}(m + m')E_{n,p}(\sigma(m + m') - \sigma(m) - \sigma(m'))E_{n,p}(\sigma(m) - m)E_{n,p}(\sigma(m') - m') \\ &\quad = F_{n,p}(m + m')E_{n,p}(\sigma(m + m') - m - m'), \end{aligned}$$

so both sides of (9.11.2) are multiplicative in m. It thus suffices to check (9.11.2) for $m = 1 = (1, 0, \dots)$, in which case $\sigma(m) = m$ and so the desired equality becomes $F_{n,p}(1) = \zeta_{p^n}$. This follows from Lemma 9.4 by evaluating at $z = \zeta_{p^n}$. □

10. Kummer-Artin-Schreier-Witt theory

In further preparation for discussion of the Oort local lifting problem, we describe a form of *Kummer-Artin-Schreier-Witt theory* for cyclic Galois extensions of a power series field.

Hypothesis 10.1. Throughout §10, fix a positive integer n, assume that K is discretely valued, the residue field k of K is algebraically closed of characteristic $p > 0$, and K contains a primitive p^n-th root of unity ζ_{p^n}. Put $F = k((\overline{z}))$.

Definition 10.2. For $\rho \in (0, 1)$, let $A(\rho, 1)$ be the annulus $\rho < |z| < 1$ in $\mathbb{P}^1_K$; the analytic functions on $A(\rho, 1)$ can be viewed as certain Laurent series in z. The union of the rings $\mathcal{O}(A(\rho, 1))$ over all $\rho \in (0, 1)$ is called the *Robba ring* over K and will be denoted $\mathcal{R}$. (This ring can be interpreted as the local ring of the adic point of $\mathbb{P}^{1,\mathrm{an}}_K$ specializing x_1 in the direction towards 0.)

Let $\mathcal{R}^{\mathrm{bd}}$ be the subring of $\mathcal{R}$ consisting of formal sums with bounded coefficients; these are exactly the elements of $\mathcal{R}$ which define *bounded* analytic functions on $A(\rho, 1)$ for some $\rho \in (0, 1)$. The ring $\mathcal{R}^{\mathrm{bd}}$ carries a multiplicative *Gauss norm* defined by

$$\left| \sum_{i \in \mathbb{Z}} a_i z^i \right| = \max_i \{|a_i|\};$$

let $\mathcal{R}^{\mathrm{int}}$ be the subring of $\mathcal{R}^{\mathrm{bd}}$ consisting of elements of Gauss norm at most 1.

Lemma 10.3. *The ring $\mathcal{R}^{\mathrm{int}}$ is a henselian discrete valuation ring. Consequently, $\mathcal{R}^{\mathrm{bd}}$ is a henselian local field with residue field F.*

Proof. See [31, Proposition 3.2]. □

We next prepare to formulate the comparison between Kummer theory and Artin-Schreier-Witt theory by introducing the two sides of the comparison.

Definition 10.4. For any field L of characteristic not equal to p, taking Galois cohomology on the exact sequence

$$1 \to \mu_{p^n} \to \overline{L}^\times \stackrel{\bullet^{p^n}}{\to} \overline{L}^\times \to 1$$

of G_L-modules gives the *Kummer isomorphism*

$$L^\times / L^{\times p^n} \cong H^1(G_L, \mu_{p^n})$$

because $H^1(G_L, \overline{L}^\times) = 0$ by Noether's form of Hilbert's Theorem 90.

In the case $L = \mathcal{R}^{\mathrm{bd}}$, by Lemma 10.3 we have a surjection $G_L \to G_F$ identifying G_F with the quotient of the maximal unramified extension of L. We thus obtain a restriction map $H^1(G_F, \mu_{p^n}) \to H^1(G_L, \mu_{p^n})$ and thus a map $H^1(G_F, \mu_{p^n}) \to L^\times/L^{\times p^n}$. Note that G_F acts trivially on μ_{p^n}, so we may identify μ_{p^n} as a G_F-module with $\mathbb{Z}/p^n\mathbb{Z}$ by identifying our chosen primitive p^n-th root of unity $\zeta_{p^n} \in \mu_{p^n}$ with $1 \in \mathbb{Z}/p^n\mathbb{Z}$. We thus end up with a homomorphism

$$H^1(G_F, \mathbb{Z}/p^n\mathbb{Z}) \to (\mathcal{R}^{\mathrm{bd}})^\times/(\mathcal{R}^{\mathrm{bd}})^{\times p^n}. \tag{10.4.1}$$

Definition 10.5. Consider the exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} = W_n(\mathbb{F}_p) \to W_n(F^{\mathrm{sep}}) \stackrel{1-\varphi}{\to} W_n(F^{\mathrm{sep}}) \to 0$$

where φ denotes the Frobenius endomorphism of $W_n(F^{\mathrm{sep}})$. The additive group $W_n(F^{\mathrm{sep}})$ is a successive extension of copies of the additive group of F^{sep}; since $H^1(G_F, F^{\mathrm{sep}}) = 0$ by the additive version of Theorem 90, we also have $H^1(G_F, W_n(F^{\mathrm{sep}})) = 0$. We thus obtain the *Artin-Schreier-Witt isomorphism*

$$\operatorname{coker}(1-\varphi, W_n(F)) \cong H^1(G_F, \mathbb{Z}/p^n\mathbb{Z}). \tag{10.5.1}$$

Combining this isomorphism with the map (10.4.1) derived from the Kummer isomorphism, we obtain a homomorphism

$$\operatorname{coker}(1-\varphi, W_n(F)) \to (\mathcal{R}^{\mathrm{bd}})^\times/(\mathcal{R}^{\mathrm{bd}})^{\times p^n}. \tag{10.5.2}$$

We note in passing how Swan conductors appear in the Artin-Schreier-Witt isomorphism.

Lemma 10.6. *For $\underline{a} \in W_n(F)$, let $\rho : G_F \to K^\times$ be the character corresponding via* (10.5.1) *to the class of $\underline{a}$ in* $\operatorname{coker}(\varphi - 1, W_n(F))$. *For $j = 0, \dots, n-1$, let $-m_j$ be the t-adic valuation of $\overline{a}_j$, and assume that m_j is not a positive multiple of p. Then the Swan conductor of ρ equals*

$$\max\{0, m_0 p^{n-1}, m_1 p^{n-2}, \dots, m_{n-1}\}.$$

In particular, if b_j is the Swan conductor of $\rho^{\otimes p^{n-j}}$, then $b_j \geq p b_{j-1}$ for $j = 1, \dots, n$.

Proof. By hypothesis, if m_j is positive then it is not divisible by p, so $m_j p^{n-1-i}$ has p-adic valuation $n-i-1$. Consequently, any two of the quantities $m_0p^{n-1}, m_1p^{n-2}, \dots, m_{n-1}$, if they are nonzero, must be distinct. It thus suffices to check the claim in case $\underline{a}$ is a Teichmüller element $[\overline{a}]$ for some $\overline{a} \in F$ of t-adic valuation $-m$ for some integer m which is positive and not divisible by p. By splitting $\overline{a}$ into powers of t, we may further reduce to the case $\overline{a} = ct^{-m}$. Using the compatibility of Swan conductors with tame base extensions, we may further reduce to the case $\overline{a} = t^{-1}$. In case $n = 1$, this case is easily checked by direct computation; the case $n > 1$ can then be reduced to this case using Herbrand's formulas relating upper ramification numbers with passage to subgroups [43, Chapter IV]. $\square$

We now make explicit the relationship between the Kummer and Artin-Schreier-Witt isomorphisms.

Theorem 10.7 (Matsuda)**.** *The homomorphism* (10.5.2) *is induced by a homomorphism*

$$W_n(\mathcal{R}^{\mathrm{int}}) \to (\mathcal{R}^{\mathrm{bd}})^\times, \qquad \underline{a} \mapsto E_{n,p}(p^n \underline{a}). \tag{10.7.1}$$

Proof. By writing

$$E_{n,p}(p^n \underline{a}) = \exp\left(\sum_{i=0}^{n-1} p^{n-i}(\zeta_{p^{n-i}} - 1)t^{p^i}\right), \tag{10.7.2}$$

we may interpret $E_{n,p}(p^n\underline{a})$ as an element of $\mathcal{R}^{\mathrm{bd}}$. By (9.7.2), the map (10.7.1) is a homomorphism. Given $\underline{a}$, choose a minimal finite separable extension S of F such that there exists $\underline{\bar{b}} \in W_n(S)$ with

$$\underline{\bar{b}} - \varphi(\underline{\bar{b}}) = \underline{\bar{a}}.$$

(This amounts to forming a tower of Artin-Schreier extensions over F.) Apply Lemma 10.3 to construct a finite étale algebra $\mathcal{S}^{\mathrm{int}}$ over $\mathcal{R}^{\mathrm{int}}$ with residue field S. Choose a lift $\underline{b} \in \mathcal{S}^{\mathrm{int}}$ of $\underline{\bar{b}}$ and put

$$f = F_{n,p}(\underline{b})E_{n,p}(\underline{a} + \sigma(\underline{b}) - \underline{b}) \in \mathcal{S}^{\mathrm{int}}.$$

Then $f^{p^n} = E_{n,p}(p^n\underline{a})$.

On one hand, the image of $\underline{a}$ in $\mathrm{coker}(\varphi - 1, W_n(F))$ corresponds to the element of $H^1(G_F, \mathbb{Z}/p^n\mathbb{Z})$ which factors through $H^1(\mathrm{Gal}(S/F), \mathbb{Z}/p^n\mathbb{Z})$ and sends $g \in \mathrm{Gal}(S/F)$ to the integer $m \in \mathbb{Z}/p^n\mathbb{Z}$ for which $\varphi(\underline{\bar{b}}) = \underline{\bar{b}} + m$. On the other hand, we have $g(\underline{b}) = \underline{b} + m + \underline{c}$ for some $\underline{c} \in W((\zeta_{p^n} - 1)\mathcal{R}^{\mathrm{int}}_n)$, and so

$$\begin{aligned}
g(f) &= F_{n,p}(\underline{b} + \underline{c} + m)E_{n,p}(\underline{a} + \sigma(\underline{b} + \underline{c} + m) - \underline{b} - \underline{c} - m) \\
&= F_{n,p}(\underline{b} + \underline{c} + m)F_{n,p}(\underline{b})^{-1}E_{n,p}(\sigma(\underline{b} + \underline{c} + m) - \sigma(\underline{b}) - \underline{c} - m)E_{n,p}(\underline{a} + \sigma(\underline{b}) - \underline{b}) \\
&= \zeta_{p^n}^m f
\end{aligned}$$

by Lemma 9.11. It follows that (10.7.1) induces (10.5.2) as desired. $\square$

Remark 10.8. By comparing a character with its p-th power, we may deduce from Theorem 10.7 that

$$\frac{E_{n,p}(p^n\underline{a})}{E_{n-1,p}(p^{n-1}\underline{a})} \in (\mathcal{R}^{\mathrm{bd}})^{\times p^{n-1}}.$$

This may also be seen directly from Lemma 9.10 by rewriting the left side as $G_{n-1,p}(\underline{a})^{p^{n-1}}$.

11. Automorphisms of a formal disc

To conclude, we use Kummer-Artin-Schreier-Witt theory to translate the Oort local lifting problem into a question about the construction of suitable connections on $\mathbb{P}^1_K$, and use this interpretation to describe existing combinatorial invariants connected with the Oort problem in terms of convergence polygons.

Hypothesis 11.1. Throughout §11, retain Hypothesis 10.1 and additionally fix $\underline{a} \in W_n(\mathcal{R}^{\mathrm{int}})$.

Definition 11.2. Let $\rho : G_F \to \mu_{p^n}$ be the character corresponding to $\underline{a}$ via the maps $W_n(\mathcal{R}^{\text{int}}) \to \text{coker}(\varphi - 1, W_n(F)) \cong H^1(G_F, \mu_{p^n})$ (the latter isomorphism being (10.5.1)). For $i = 1, \dots, n$, let b_i be the Swan conductor of $\rho^{\otimes p^{n-i}}$.

For $x_1 \in \mathbb{P}^{1,\text{an}}_K$ as usual, we may lift ρ uniquely to an unramified character $\tilde{\rho} : G_{\mathcal{H}(x_1)} \to \mu_{p^n}$. By Katz's theory of canonical extensions (in this context see [31]), this character arises from a finite étale Galois cover of a subspace of $\mathbb{P}^{1,\text{an}}_K$ of the form $|z| > \rho$ for some $\rho \in (0, 1)$. We may then proceed as in Definition 8.2 to obtain a rank 1 bundle $\mathcal{E}_n$ with connection on this subspace; this can be described explicitly as the free vector bundle on a single generator $\mathbf{v}$ equipped with the connection

$$\nabla(\mathbf{v}) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} (\zeta_{p^{n-j}} - 1) a_i^{p^j - 1} \mathbf{v} \otimes da_i$$

Formally, we have $d\mathbf{v} = \mathbf{v} \otimes d \log E_{n,p}(\underline{a})$.

For $i = 1, \dots, n$, put $\mathcal{E}_i = \mathcal{E}_n^{\otimes p^{n-i}}$. Then $\mathcal{E}_i$ corresponds to the character $\rho^{\otimes p^{n-i}}$ of order p^i in a similar fashion.

Definition 11.3. Let S be the fixed field of $\ker(\rho)$; we may identify S with $k((\overline{u}))$ for some parameter $\overline{u}$. The action of G_F defines a k-linear automorphism of $k((\overline{u}))$ of order p^n, which induces an automorphism τ of $k[\![\overline{u}]\!]$. A *solution of the lifting problem* for ρ is a lifting of τ to an $\mathfrak{o}_K$-linear automorphism $\tilde{\tau}$ of $\mathfrak{o}_K[\![u]\!]$.

Conjecture 11.4 (Oort). *A solution of the lifting problem exists for every ρ.*

A spectacular breakthrough on this problem has been made recently in work of Obus–Wewers and Pop [32, 35].

Theorem 11.5. *For fixed ρ, a solution of the lifting problem exists over some extension of K.*

We will not say anything more here about the techniques used to prove Theorem 11.5. Instead, we describe an equivalence between solutions of the lifting problem for ρ and extensions of the connection on $\mathcal{E}_n$.

Definition 11.6. Suppose that $\tilde{\tau}$ is a solution of the lifting problem. Then $\tilde{\tau}$ gives rise to a finite étale cover of the disc $|z| < 1$, which may be glued together with the cover from Definition 11.2 to give a finite cover $f_n : Y_n \to X$ with $X = \mathbb{P}^1_K$, such that x_1 has a unique preimage. For $i = 1, \dots, n$, let $f_i : Y_i \to X$ be the cover corresponding to $\rho^{p^{n-i}}$ in similar fashion, and let Z_i be the ramification locus of f_i; also put $Z_0 = \emptyset$. By the Riemann-Hurwitz formulas in characteristic 0 and p, we have

$$\begin{aligned} 2 - 2g(Y_i) &= 2p^i - \sum_{j=1}^{i} (p^i - p^{j-1})(\text{length}(Z_j) - \text{length}(Z_{j-1})) \\ &= 2p^i - \sum_{j=1}^{i} (p^j - p^{j-1}) b_j. \end{aligned}$$

Solving these equations yields

$$\text{length}(Z_i) = b_i + 1 \qquad (i = 1, \dots, n).$$

Let $\mathcal{N}_i$ be the convergence polygon of $\mathcal{E}_i$. For each $x \in Z_i - Z_{i-1}$, $\mathcal{E}_n$ is regular at x with exponent m/p^{n-i-1} for some $m \in \mathbb{Z} - p\mathbb{Z}$; by Theorem 6.2, $\mathcal{N}_n$ is constant in some neighborhood of x. Using similar techniques, we can see that for y in some neighborhood of x,

$$s_1(\mathcal{N}(x)) = \frac{p}{p-1} \log p.$$

Using Theorem 7.23, we may deduce that the graph Γ can be taken to be the union of the paths from the elements of Z to x_1. Moreover, for each $x \in \Gamma - \{x_1\}$, for $\vec{t}$ the branch of x towards x_1, $\partial_{\vec{t}} = m - 1$ where m is the length of the subset of Z dominated by x. Note that the continuity of $\mathcal{N}$ now imposes some combinatorial constraints on the relative positions of the elements of Z; these constraints encode the data of the *Hurwitz tree* associated to $\tilde{\tau}$ [1].

Definition 11.7. Conversely, suppose $X = \mathbb{P}^1_K$; Z is contained in the open unit disc; $\mathcal{E}$ is a rank 1 vector bundle with connection on $U = X - Z$; for each $z \in Z$, ∇ is regular at z with exponent in $p^{-n}\mathbb{Z}$; and for some $\rho \in (0,1)$, the restriction of $\mathcal{E}$ to the space $|z| > \rho$ is isomorphic to $\mathcal{E}_n$. Using Dwork's transfer theorem, we deduce that the connection on $\mathcal{E}^{\otimes p^n}$ is globally trivial; consequently, $\mathcal{E}$ corresponds to a solution of the lifting problem.

Remark 11.8. For a given ρ, it should be possible to construct a moduli space of solutions of the lifting problem in the category of rigid analytic spaces over K. Theorem 11.5 would then imply that this space is nonempty. Given this fact, it may be possible to derive additional results on the lifting problem, e.g., to resolve the case of dihedral groups. For $p > 2$, this amounts to showing that if τ anticommutes with the involution $\overline{z} \mapsto -\overline{z}$, then the action of the involution $z \mapsto -z$ fixes some point of the moduli space.

Appendix A. Monotonicity in a disc

In this appendix, we give some technical arguments needed for the proof of Theorem 6.5 in the case $p > 0$, which do not appear elsewhere in the literature.

Definition A.1. For I a subinterval of $[0, +\infty)$, let R_I be the ring of rigid analytic functions on the space $|t| \in I$ within the affine t-line over K, as in [28, Definition 3.1.1].

Hypothesis A.2. Throughout Appendix A, assume $p > 0$, and let (M, D) be a differential module of rank n over $R_{[0,\beta)}$.

Definition A.3. For I a subinterval of $[0, \beta)$, write M_I as shorthand for $M \otimes_{R_{[0,\beta)}} R_I$. For $r > -\log \beta$ and $i = 1, \dots, n$, we define the functions $f_i(M, r)$ and $F_i(M, r)$ as in [25, Notation 11.3.1].

Lemma A.4. *Assume $p > 0$. Suppose that M admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Choose $\alpha \in (0, \beta)$. Suppose that $M_{[\alpha,\beta)}, M^\vee_{[\alpha,\beta)}$ admit differential submodules free on the single respective generators $\mathbf{v}, \mathbf{v}^\vee$, and that*

$$D(\mathbf{v}) = \lambda t^{-1} \mathbf{v}, D(\mathbf{v}^\vee) = -\lambda t^{-1} \mathbf{v}^\vee, \langle \mathbf{v}, \mathbf{v}^\vee \rangle = 1$$

for some $\lambda \in \mathfrak{o}_K$ (if $p = 0$) or $\lambda \in \mathbb{Z}_p$ (if $p > 0$). If

$$\langle \mathbf{v}, \mathbf{e}_i^\vee \rangle \langle \mathbf{e}_j, \mathbf{v}^\vee \rangle \in R_{[0,\beta)} \qquad (i, j = 1, \dots, n),$$

then $\lambda \in \mathbb{Z}$, $t^{-\lambda}\mathbf{v} \in H^0(M)$, $t^\lambda \mathbf{v}^\vee \in H^0(M^\vee)$.

Proof. By the given hypothesis, the intersection within $M_{[\alpha,\beta)}$ of M with the $R_{[\alpha,\beta)}$-span of $\mathbf{v}$ is a nonzero differential submodule M_0 of rank 1. By [25, Example 9.5.2], we have $f_1(M_0, r) = 0$ for $-\log\delta \leq r \leq -\log\gamma$. Choose $\epsilon \in (\gamma, \delta)$; by the Dwork transfer theorem [25, Theorem 9.6.1], M_0 admits a nonzero horizontal element $\mathbf{w}$. Write $\mathbf{v} = c\mathbf{w}$ for some $c \in R_{[\alpha,\beta)}$, then write $c = \sum_{i\in\mathbb{Z}} c_i t^i$; the equality $D(\mathbf{v}) = \lambda t^{-1}\mathbf{v}$ implies that $ic_i = \lambda c_i$ for all i. Since $\mathbf{v} \neq 0$, we must have $\lambda \in \mathbb{Z}$ and $t^{-\lambda}\mathbf{v} \in M$; similarly, $t^{\lambda}\mathbf{v}^\vee \in M^\vee$. □

Lemma A.5. *Assume $p > 0$. Suppose that M admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Choose $\alpha \in (0, \beta)$. Let $N \oplus P$ be a direct sum decomposition of $M_{[\alpha,\beta)}$ as a differential module such that* $\operatorname{rank}(N) = m$, *and use this decomposition to identify $N^\vee$ with a submodule of $M^\vee_{[\alpha,\beta)}$. Suppose that there exist $\mathbf{v} \in \wedge^m N_{[\alpha,\beta)}$, $\mathbf{v}^\vee \in \wedge^m N^\vee_{[\alpha,\beta)}$ satisfying*

$$D(\mathbf{v}) = \lambda t^{-1}\mathbf{v}, D(\mathbf{v}^\vee) = -\lambda t^{-1}\mathbf{v}^\vee, \langle \mathbf{v}, \mathbf{v}^\vee\rangle = 1$$

for some $\lambda \in \mathfrak{o}_K$ (if $p = 0$) or $\lambda \in \mathbb{Z}_p$ (if $p > 0$). If for all $1 \leq i_1 < \cdots < i_m \leq n$, $1 \leq j_1 < \cdots < j_m \leq n$ we have

$$\langle \mathbf{v}, \mathbf{e}^\vee_{i_1} \wedge \cdots \wedge \mathbf{e}^\vee_{i_m}\rangle \langle \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_m}, \mathbf{v}\rangle \in K\langle t/\delta\rangle,$$

then $N, N^\vee$ descend to respective differential submodules of $M, M^\vee$.

Proof. Immediate from Lemma A.4. □

Lemma A.6. *Assume $p > 0$. Suppose that $\beta > 1$ and that $H^0(M_{[0,\delta)}) = 0$ for all $\delta \in (1, \beta)$. Then for $i = 1, \dots, n$, the left slope of $F_i(M, r)$ at $r = 0$ is nonpositive.*

Proof. There is no harm in reducing β, so by [25, Theorem 11.3.2(a)] we may ensure that for $i = 1, \dots, n$, $f_i(M, r)$ is affine for $-\log\beta < r \leq 0$. Since $R_{[0,\delta]}$ is a principal ideal domain for all $\delta > 0$, we may also ensure that M admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

Choose $m \in \{0, \dots, n\}$ such that for $i \in \{1, \dots, n\}$, $i > n - m$ if and only if $f_i(M, r) = r$ for r in some left neighborhood of 0. We proceed by induction on m. If $m = 0$, then the claim follows immediately from [25, Theorem 11.3.2(d)]; we may thus assume $m > 0$ hereafter.

By [28, Lemma 3.7.3], $M_{(1,\beta)}$ admits a spectral decomposition $M_0 \oplus M_1 \oplus \cdots$ in which M_0 is the Robba component; in particular, $\operatorname{rank}(M_0) = m$. By [25, Theorem 13.6.1], we can further ensure that $\wedge^m M_0$ admits a generator $\mathbf{v}$ such that $D(\mathbf{v}) = \lambda t^{-1}\mathbf{v}$ for some $\lambda \in \mathbb{Z}_p$. Using the direct sum decomposition of $M_{(1,\beta)}$, we may view the dual generator $\mathbf{v}^\vee$ of $\wedge^m M_0^\vee$ as an element of $M^\vee_{(1,\beta)}$.

Note that M_0 cannot descend to a differential submodule N of M, as otherwise the Dwork transfer theorem [25, Theorem 9.6.1] would imply that N is trivial and hence $H^0(M) \neq 0$. Consequently, Lemma A.5 implies the existence of $1 \leq i_1 < \cdots < i_m \leq n$, $1 \leq j_1 < \cdots < j_m \leq n$ such that for

$$\begin{aligned} c_{i_1,\dots,i_m} &= \langle \mathbf{v}, e^\vee_{i_1} \wedge \cdots \wedge e^\vee_{i_m}\rangle \\ c^\vee_{j_1,\dots,j_m} &= \langle e_{j_1} \wedge \cdots \wedge e_{j_m}, \mathbf{v}^\vee\rangle, \end{aligned}$$

we have $c_{i_1,\dots,i_m} c^\vee_{j_1,\dots,j_m} \notin R_{[0,\beta)}$. We can thus choose a nonnegative integer h such that the coefficient of t^{-1} in $t^h c_{i_1,\dots,i_m} c^\vee_{j_1,\dots,j_m}$ is a nonzero element $\mu \in K$.

For $\nu \in K$, let (M_ν, D_ν) be the differential module on the same underlying module as M, but with

$$D_\nu(\mathbf{e}_i) = \begin{cases} D(\mathbf{e}_i) + \nu t^h \mathbf{e}_{j_1} & i = i_1 \\ D(\mathbf{e}_i) + \nu \mathbf{e}_{j_k} & i = i_k \quad (k = 2, \dots, m) \\ D(\mathbf{e}_i) & \text{otherwise.} \end{cases}$$

Choose γ, δ with $1 < \gamma < \delta < \beta$. Using [25, Corollary 6.5.4, Theorem 10.5.1], we see that for any $\epsilon > 0$, for ν sufficiently small,

$$f_i(M_\nu, r) \begin{cases} = f_i(M, r) & (i = 1, \dots, n-m) \\ \leq r + \epsilon & (i = n-m+1, \dots, n) \end{cases} \qquad (r \in [-\log \delta, -\log \gamma]).$$

Suppose that there exist $\nu \in K$ arbitrary close to 0 such that $f_{n-m+1}(M_\nu, -\log \gamma) > -\log \gamma$. Let $g(\nu)$ be the slope of the segment in $\mathbb{R}^2$ from $(-\log \delta, F_i(M_\nu, \log \delta))$ to $(-\log \gamma, F_i(M_\nu, \log \gamma))$. For $i = 1, \dots, n$, by [25, Theorem 11.3.2(e)] the function $F_i(M_\nu, r)$ is convex, so $g(\nu)$ is no greater than the left slope of $F_i(M_\nu, r)$ at $r = -\log \gamma$. By the induction hypothesis, it follows that $g(\nu) \leq 0$; by taking the limit as $\nu \to 0$, we deduce the desired result.

It thus remains to derive a contradiction under the contrary assumption: for ν sufficiently small, $f_{n-m+1}(M_\nu, -\log \gamma) = -\log \gamma$. By a similar argument, we may also assume that for ν sufficiently small, $f_{n-m+1}(M_\nu, -\log \delta) = -\log \delta$. Since $F_{n-m+1}(M_\nu, r)$ is convex for all i and $f_i(M_\nu, r) = f_i(M, r)$ is affine for $i = 1, \dots, n-m$, it follows that $f_{n-m+1}(M_\nu, r) = r$ for $r \in [-\log \delta, -\log \gamma]$. Consequently, $M_{\nu,(\gamma,\delta)}$ also admits a spectral decomposition in which the Robba component $M_{\nu,0}$ has rank m. By tracing through the proof of [25, Theorem 12.4.1], we see that $\wedge^m M_{\nu,0}$ admits a generator of the form $\mathbf{v} + \nu \mathbf{w}$, where $\mathbf{w}$ is a locally analytic function of ν. We may further choose $\mathbf{w}$ so that the action of D' on $\mathbf{v} + \nu \mathbf{w}$ is given by multiplication by $\lambda + \nu \xi t^{-1}$, where $\xi \in K$ is a locally analytic function of ν with $\xi(0) = \mu \neq 0$. We may thus arrange for $\nu \xi$ to equal any sufficiently small value in K, and in particular a value not in $\mathbb{Z}_p$; however, the fact that $\wedge^m M_{\nu,0}$ satisfies the Robba condition now contradicts [25, Example 9.5.2]. We now have the desired contradiction, completing the proof. □

Definition A.7. Let $\mathbb{D}_\beta$ be the Berkovich disc $|t| < \beta$ over K. For $x \in \mathbb{D}_\beta$, define the real numbers $s_i(M, x)$ for $i = 1, \dots, n$ as in [28, Definition 4.3.2]; note that they are invariant under extension of K [28, Lemma 4.3.3].

Lemma A.8. *For $\rho \in (0, \beta)$, let x_ρ be the generic point of the disc $|t| \leq \rho$. Then for $i = 1, \dots, n$, the function $\rho \mapsto s_1(M, x_\rho) \cdots s_i(M, x_\rho)$ is nonincreasing in ρ.*

Proof. For $r > -\log \beta$, define the functions

$$\begin{aligned} g_i(M, r) &= -\log s_i(M, x_{e^{-r}}) \\ G_i(M, r) &= g_1(M, r) + \cdots + g_i(M, r). \end{aligned}$$

By [28, Theorem 4.5.15], the functions $g_i(M, r)$ are continuous and piecewise affine; it thus suffices to check that the right slope of $G_i(M, r)$ is nonpositive for $i = 1, \dots, n$. By [28, Proposition 4.3.7], we have

$$f_i(M, r) = \max\{r, g_i(M, r)\} \qquad (i = 1, \dots, n).$$

Put $m = \dim_K H^0(M)$. then $s_i(M, x) = \beta$ for $i = n-m+1, \dots, n$ and $x \in \mathbb{D}$. Moreover, if we write N for the span of $H^0(M)$, then $s_i(M, x) = s_i(M/N, x)$ for $i = 1, \dots, n-m$ and $x \in \mathbb{D}$. Consequently, we may reduce to the case where $H^0(M) = 0$; this implies in turn that $H^0(M_{[0,\delta)}) = 0$ for all $\delta \in (0, \beta]$ sufficiently close to β. Consequently, for $r \in (-\log \beta, -\log \delta)$ we cannot have $g_i(M, r) < r$ (see [28, Remark 4.3.4]) so we must in fact have $g_i(M, r) = f_i(M, r)$. We may thus deduce the claim from Lemma A.6. □

Theorem A.9. *For $x, y \in \mathbb{D}_\beta$ such that x dominates y, for $i = 1, \dots, n$, we have*

$$s_1(M, x) \cdots s_i(M, x) \leq s_1(M, y) \cdots s_i(M, y).$$

In particular, the conclusion of Theorem 6.5 holds for $p > 0$.

Proof. This follows from Lemma A.8 thanks to the invariance of the s_i under base extension. □

Appendix B. Convexity

In this appendix, we give some additional technical arguments needed for the proof of Theorem 6.5 in the case $p = 0$ and the proof of Theorem 7.23(c), which do not appear elsewhere in the literature.

Hypothesis B.1. Throughout Appendix B, assume $p = 0$, and let (M, D) be a differential module of rank n over $R_{[0,\beta)}$.

We begin with a variant of Theorem 7.23.

Lemma B.2. *The right slope of $G_i(M, r)$ at $r = -\log \beta$ equals $-\dim_K H^1(M)$, provided that at least one of the two is finite.*

Proof. Apply [34, Theorem 3.5.2]. □

This gives an analogue of Lemma A.8 in the case $p = 0$.

Lemma B.3. *For $\rho \in (0, \beta)$, let x_ρ be the generic point of the disc $|t| \leq \rho$. Then for $i = 1, \dots, n$, the function $\rho \mapsto s_1(M, x_\rho) \cdots s_i(M, x_\rho)$ is nonincreasing in ρ.*

Proof. The case $i = n$ is immediate from Lemma B.2. To deduce the case $i < n$, it suffices to work locally around some $\rho_0 = e^{-r_0}$, and to check only those values of i for which the left slopes of $g_i(M, r_0) > g_{i+1}(M, r_0)$. For $\lambda \in K$, let M_λ be the differential module obtained from M by adding λ to D. Since $G_i(M, r)$ is insensitive to extension of the field K, we may ensure that in some neighborhood of r_0, we have $g_j(M, r) = g_j(M_\lambda, r)$ for $j \leq i$ while $g_j(M_\lambda, r)$ is constant for $j > i$. We thus deduce the claim from the case $i = n$. □

This yields the analogue of Theorem A.9 for $p = 0$.

Theorem B.4. *For $x, y \in \mathbb{D}_\beta$ such that x dominates y, for $i = 1, \dots, n$, we have*

$$s_1(M, x) \cdots s_i(M, x) \leq s_1(M, y) \cdots s_i(M, y).$$

In particular, the conclusion of Theorem 6.5 holds for $p > 0$.

Proof. This follows from Lemma B.3 thanks to the invariance of the s_i under base extension. □

We next proceed towards Theorem 7.23(c).

Definition B.5. Define the convergence polygon $\mathcal{N}$ of M as in Definition 5.3, using the same disc at every point. Define the modified convergence polygon $\mathcal{N}'$ using maximal discs not containing 0.

Lemma B.6. *Assume $p = 0$. For $i = 1, \dots, n$, for $x \in \mathbb{D}_\beta$, we have $\Delta h_i(\mathcal{N})_x \geq 0$.*

Proof. By base extension, we may reduce to the case $x = x_1$. As in the proof of Lemma B.3, we may further reduce to the case $i = n$. We may further reduce to the case where $H^0(M_{[0,\delta)}) = 0$ for all $\delta \in (1, \beta)$.

Choose a nonempty set W of K-rational points of $\mathbb{D}_\beta$ with the following properties.

(a) For each $w \in W$, $U_{w,1}$ is a branch of X at x_1, which we also denote by $\vec{t_w}$.
(b) The branches $\vec{t_w}$ for $w \in W$ are pairwise distinct.
(c) Let $\vec{t_\infty}$ be the branch of X at x in the direction of $\Gamma_{X,Z}$. Then for all branches $\vec{t}$ of X at x_1, we have $\partial_{\vec{t}}(\mathcal{N}) = 0$ unless $\vec{t} = \vec{t_\infty}$ or $\vec{t} = \vec{t_w}$ for some $w \in W$.

Let $R_{w,I}$ be the ring of rigid analytic functions on the space $|z - w| \in I$, and put $M_{w,I} = M \otimes_{R_{[0,\beta)}} R_{w,I}$. Then by Lemma B.2, to prove the desired result, it suffices to check that for any $\gamma \in (0, 1)$, $\delta \in (1, \beta)$ sufficiently close to 1,

$$\dim_K H^1(M_{[0,\delta)}) \geq \sum_{w \in W} \dim_K H^1(M_{w,[0,\gamma)}).$$

It would hence also suffice to prove surjectivity of the map

$$H^1(M_{[0,\delta)}) \to \sum_{w \in W} H^1(M_{w,[0,\gamma)}),$$

or equivalently of the map

$$H^1(M_{(\gamma',\delta)}) \to \bigoplus_{w \in W} H^1(M_{w,(\gamma',\gamma)}) \tag{B.6.1}$$

for some $\gamma' \in (0, \gamma)$. By choosing γ' sufficiently close to γ, we may apply [28, Lemma 3.7.6] to see that $M_{w,(\gamma',\gamma)} \to H^1(M_{w,(\gamma',\gamma)})$ is a strict surjection of Fréchet spaces, where $H^1(M_{w,(\gamma',\gamma)})$ carries the usual topology on a finite-dimensional K-vector space. Since the map

$$M_{(\gamma',\delta)} \to \bigoplus_{w \in W} M_{w,(\gamma',\gamma)}$$

has dense image, so then does (B.6.1), proving the claim. □

Theorem B.7. *The conclusion of Theorem 7.23(c) holds.*

Proof. By Theorem 7.23(b), we may assume $x \notin \Gamma_{X,Z}$; the claim thus reduces to Lemma B.6. □

Remark B.8. It may be possible to adapt the proof of Theorem B.7 to the case $p > 0$ by establishing a relative version of Lemma B.2. In the notation of the proof of Lemma B.6, one might hope to prove that

$$H^1(M_{(\gamma',\delta)} / \bigoplus_{w \in W} M_{w,(\gamma',\gamma)}) = 0$$

and to relate this vanishing to the Laplacian, bypassing the potential failure of finite-dimensionality by the individual cohomology groups.

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