PROBLEM SET 1 - DHAR'S BURNING ALGORITHM AND RIEMANN–ROCH

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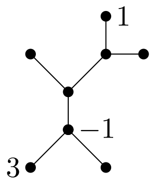
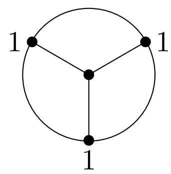
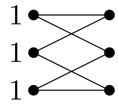
The graphs in this problem set are all finite, but notice that metric graphs would only require minor changes.

Warm up questions.

(1) Compute the rank of the following divisors.

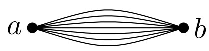






(2) Let G be the banana graph consisting of two vertices and $g+1$ edges between them. Let D be the divisor with a chips on one of the vertices and b chips on the other, where $0 \leq a \leq b$. Show that

$$r(D) = \begin{cases} a & b \leq g \\ a+b-g & \text{otherwise} \end{cases}$$



(3) Let D be a v-reduced divisor such that $D(v) = a$. Show that $r(D) \leq a$. Find an example where $r(D)$ is strictly smaller than a.

(4) Let G be a trivalent graph of genus g with E edges and V vertices.
- Express E in terms of g.
- Let D be a divisor of degree $E \cdot V$. Show that there must be a vertex v, whose valency is smaller or equal $D(v)$. Conclude that every reduced divisor of degree $E \cdot V$ is effective.
- Prove that every divisor of degree $E \cdot V + r$ has rank at least r.

(5) Use Riemann–Roch to show that the rank of every divisor of degree $d > 2g-2$ is $d-g$. Show that a divisor of degree $2g-2$ has rank $g-1$ if and only if it is equivalent to the canonical divisor.

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Dhar's burning algorithm. Let D be a divisor on a finite graph G, and fix a vertex q on the graph.

(1) Show that D is equivalent to a divisor that is effective everywhere away from q (but may be very negative at q).

If the divisor is not reduced, then by definition there is a set of vertices which does not contain q, such that firing from them does not create a pole. Fire them, and check whether we get a reduced divisor.

(2) Show that after repeating this finitely many times, we end up with a reduced divisor.

(3) Let A be the set where a rational function f obtains its minimum, and let $\mathrm{div}(A)$ the divisor obtained by firing each of the vertices of A once. Show that $\mathrm{div}(f)(v) \leq \mathrm{div}(A)(v)$ at very vertx v of A. Conclude that every divisor is equivalent to a *unique* reduced divisor.

Riemann–Roch. The following notations and definitions will be useful throughout:

- For a rational function f, we have $\mathrm{div}(f)(v) = \sum(f(v) - f(u))$, where the sum is taken over the vertices adjacent to v. In other words, the divisor $\mathrm{div}(f)$ is obtained by chip-firing $-f(v)$ times from each vertex v.
- For a set A of vertices, the divisor $\mathrm{div}(A)$ is the principal divisor obtained by firing once from each vertex of A.
- Given an orientation $\mathcal{O}$, there is an associated divisor $D_{\mathcal{O}}$ whose degree at every vertex is the number of incoming edges minus 1. A divisor of the form $D_{\mathcal{O}}$ is called a *moderator*.
- A *source* is a vertex for whom all the adjacent edges are oriented outwards.
- An orientation is *acyclic* if it doesn't have any oriented cycles.
- Given a divisor D, we define $\deg^+(D)$ as the sum of the non-negative coefficients of D.

(1) Show that for an orientation $\mathcal{O}$, the divisor $D_{\mathcal{O}}$ has degree $g - 1$.

(2) Show that $D_{\mathcal{O}} + D_{\mathcal{O}^-}$ is equal to the canonical divisor, where $\mathcal{O}^-$ represents the reverse orientation of $\mathcal{O}$.

From now on, we fix an acyclic orientation $\mathcal{O}$.

(3) Show that $D_{\mathcal{O}}$ is not equivalent to an effective divisor:
   - Assume by contradiction that $D_{\mathcal{O}} + \mathrm{div}(f)$ is effective, and let A be a connected component of the set on which f obtains its minimum.
   - Show that $(D_{\mathcal{O}} + \mathrm{div}(f))|_A \leq (D_{\mathcal{O}} + \mathrm{div}(A))|_A$.
   - Show that the restriction of $\mathcal{O}$ to A has a source.
   - Conclude that $D_{\mathcal{O}} + \mathrm{div}(f)$ is not effective.

Let D be a q-reduced divisor with respect to some vertex q. When starting a fire from q (in the sense of the burning algorithm) it burns through the whole graph. At each vertex, choose a direction in which the fire spreads. Fixing such a choice determines an acyclic orientation $\mathcal{O}$, and a moderator $\nu = D_{\mathcal{O}}$.

(4) Show that $D(v) \leq D_{\mathcal{O}}(v)$ for every $v \neq q$, and conclude that either D or $\nu - D$ is effective.

(5) Show that for every moderator ν associated to an orientation obtained as above, the rank of D is strictly smaller than $\deg^+(D - \nu)$.
(6) In fact, show that the rank of D equals $m - 1$, where m is the minimum of $\min(\deg^+ D' - \nu)$ as ν varies over all the moderators, and D' varies over the divisor class of D.
(7) Prove the Riemann–Roch theorem, namely,

$$r(D) - r(K - D) = \deg(D) - g + 1.$$