PROBLEM SET 2 - SPECIAL DIVISORS ON CHAIN OF LOOPS

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In this worksheet, we will largely be interested in the so-called "chain of g loops", depicted in the figure below. Our goal will be to establish a correspondence between Young tableaux and special divisors on the chain.

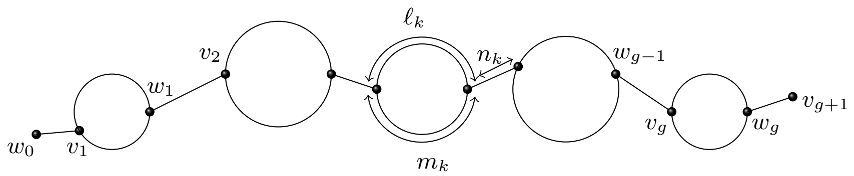

FIGURE 1. The chain of g loops.

(1) Show that the chain is hyperelliptic if and only if $m_i = \ell_i$ for $i = 2, \ldots, g-1$.
(2) Let Γ be a chain of 4 loops with generic edge lengths (that is, $\frac{m_i}{\ell_i} \in \mathbb{R} \setminus \mathbb{Q}$). Up to equivalence, there are precisely two divisors of degree 3 and rank 1. Find them. *Hint:* Try to first find a configuration of two chips to the left of w_1, such that at least one chip can be moved to the third loop.
(3) What happens when the lengths above are not generic? For instance, if the chain is hyperelliptic, what are the divisors of degree 3 and rank 1?
(4) Let Γ be the loop of 3 loops.

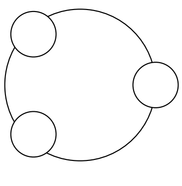

FIGURE 2. The chain of 3 loops.

How many g^1_3's are there? The answer may or may not be finite and may depend on the edge length.

Let Γ be a chain of g loops. Fix integers d and r, denote $h = g - d + r$, and let

$$\rho(g, r, d) = g - (r+1)h.$$

Date: July 29, 2016.

Generic edge length, $\rho = 0$. By *generic* we mean that the ratio between the upper and lower arc of each loop is an irrational number. Consider a Young tableaux with $r+1$ columns and h rows, filled with integers $1, \dots, g$ (confirm that since $\rho = 0$, the number of boxes equals g). Define a divisor D_λ as follows. Let $i \in \{1, \dots, g\}$. If it appears on the rightmost column of λ, then the i-th loop will not contain a chip. Otherwise, let c_i be the column number that contains i (counted from the left), d_i the row number (counted from the top), and e_i the number indices smaller than i on the rightmost column. Now, place a chip on the i-th loop at distance $(r - c_i + d_i - e_i) \cdot m_i$ counter clockwise from the rightmost vertex of the loop. Place the remaining chips (confirm that there are precisely r of them when $\rho = 0$) at the leftmost vertex.

(1) Find the divisors associated with the following tableaux:

1	2
3	4

1	4	7
2	5	8
3	6	9

1	2	$\cdots$	g

(Hint: you don't actually need to follow the construction for the last case).

(2) Show that for each $k = 0, \dots, r$, the divisor D_λ has a representative with $r - k$ chips at w_0 and k chips at v_{g+1}.
(3) Prove that D_λ as constructed above has degree d and rank exactly r.
(4) Prove that different tableaux yield non-equivalent divisors.
(5) Show that every g^r_d corresponds to a tableaux as above. Hint: consider the w_0 reduced divisor, and start passing chips from left to right.
(6) Given a tableaux λ, which tableaux corresponds to the divisor $K - D_\lambda$?
(7) Show that there are exactly

$$g! \prod_{i=0}^{r} \frac{i!}{(h+i)!}$$

divisor classes of degree d and rank r.
(8) Show that when $\rho < 0$ there are no divisors of degree d and rank r.

Generic edge length, $\rho > 0$. In this case, the genus is strictly larger than the number of boxes. Construct a divisor D_λ by placing chips on loops as before when an index appears in the tableaux, and a chip at an arbitrary position on a loop corresponding to an index that does not appear in the tableaux.

(1) Check that the number of chips placed at w_0 is still r.
(2) Show that D_λ has rank r.
(3) Show that the set of divisors of degree d and rank r has dimension ρ (we will not provide a precise description to the space of g^r_d's, but there is a very natural way of doing that).

Any edge length. Let t_i be the torsion order of m_i in the group $\mathbb{Z}/(\ell_i + m_i)\mathbb{Z}$. In other words, it is the smallest integer such that $t_i \cdot m_i$ is a multiple integer of the length of the entire loop, and is 0 if such an integer doesn't exist. Consider a tableaux in which numbers are allowed to repeat, as long as the following rule is preserved: if $i = \lambda(a, b) = \lambda(c, d)$ then $a - b \equiv c - d \mod t_i$, and construct a divisor D_λ following the recipe above.

(1) Show that a divisor constructed this way is well defined.
(2) Find edge lengths for which the following tableaux are admissible, and construct the corresponding divisors:

1	2
2	3

1	2	$\cdots$	$g-1$
2	3	$\cdots$	g

(3) Show that D_λ has rank r.