PROBLEM SET 3 - THE SPECIALIZATION LEMMA

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Let K be a complete discretely valued field with valuation ring R and residue field k. Assume k is algebraically closed. Let X be a smooth, proper, geometrically connected curve over K. Throughout, we assume that $\mathscr{X} \to Spec(R)$ is a strongly semistable model for X. That is, $\mathscr{X}$ is regular, flat and proper over R, has generic fiber identified with X, and has reduced special fiber, with smooth components and ordinary double point singularities. Let $C_1, \ldots, C_n$ be the components of $\mathscr{X}_k$, and $v_1, \ldots, v_n$ the corresponding vertices of the dual graph of $\mathscr{X}_k$.

Justify the following facts.

(1) Every Weil divisor on X (or on $\mathscr{X}$) is Cartier.
(2) Given a line bundle $\mathscr{L}$ on $\mathscr{X}$, the degree of $\mathscr{L}$ restricted to X is equal to the degree of $\mathscr{L}$ restricted to $\mathscr{X}_k$.
(3) The generic fiber is dense in the total family.

Given a divisor $\mathscr{D}$ on $\mathscr{X}$, we obtain a divisor $\rho(\mathscr{D})$, the specialization of $\mathscr{D}$, on the dual graph G of $\mathscr{X}_k$ by setting

$$\rho(\mathscr{D}) = \sum_{i=1}^{n} (\mathscr{D} \cdot C_i) v_i.$$

Prove the following.

(1) If $\mathscr{D}$ is a principal divisor, then $\rho(\mathscr{D}) = 0$.
(2) If $\mathscr{D}$ is supported on the special fiber, then $\rho(\mathscr{D})$ is a principal divisor on G.

Given a divisor D on X, the scheme theoretic closure (extended linearly) can be specialized to G giving a map

$$\mathrm{Div}(X) \to \mathrm{Div}(G).$$

Verify the following statements.

(1) If D is effective on X, then $\rho(D)$ is effective.
(2) For any divisor D on X, $\deg(D) = \deg(\rho(D))$.
(3) If D is principal on X, then $\rho(D)$ is principal.

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Specializing Points

(1) Given a smooth point $\overline{x} \in \mathscr{X}_k$, prove that there exists a K-point x in X specializing to it. *Hint: You may request the help of a grandson of German pianist and composer Fanny Mendelssohn.*
(2) Deduce that for all vertices in G, there exists a K-point of X specializing to it.
(3) Deduce that X has infinitely many K-points.

Prove Matt's Specialization Lemma: For any divisor D on X, the rank of $\rho(D)$ is at least the rank of D.

Hint: You wish to show that you can subtract r points from the specialization of D and make it equivalent to effective. Choose a degree r effective divisor on G and lift it to an effective divisor on X.

Take as a blackbox that every discrete graph without loops is the dual graph of some strongly semistable model as above. Give examples to show that this inequality can be strict.