PROBLEM SET 4 - RANK DETERMINING SETS

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A subset R of a metric graph Γ is said to be *rank determining* if the rank of every divisor can be verified by only considering points of R. That is, a divisor D has rank r when $|D - E| \neq \emptyset$ for every effective divisor E of degree r supported on R. The goal of the exercise is the following result.

Theorem. *Let Γ be a metric graph with a chosen loopless model. Then the vertices of Γ in this model form a rank determining set. In addition, if the genus of Γ is g, then Γ has a rank determining set of size $g + 1$.*

Let A be a closed subset of Γ. The *out-degree* of a point p from A, denoted $\text{outdeg}_A(p)$ is the number of tangent directions emanating from p into the complement of A. A boundary point p of A is *saturated* with respect to a divisor D if $\text{outdeg}_A(p) \leq D(p)$. Recall that a divisor D is said to be *reduced* with respect to a point q if it satisfies the following.

- D is effective away from q.
- Every closed, connected set $A \subseteq \Gamma \backslash \{q\}$ has a boundary point that is not saturated with respect to D.

Finally, recall that for every divisor D and a point q there exists a unique q-reduced divisor equivalent to D, denoted D_q. An open set is said to be a *YL set* if every connected component of its complement contains a boundary point of out-degree at least 2.

(1) Show that a subset A is rank-determining if and only if for every divisor D having rank -1 and any $q \in \Gamma$, there exists a point $a \in A$ such that the divisor $D + q - a$ has rank -1.
(2) Show that a subset A of Γ is rank determining if and only if for any $q \in \Gamma$ and any acyclic orientation $\mathcal{O}$ with a unique source at q, there is a point $a \in A$ such that $D_{\mathcal{O}} + q - a$ has rank -1.
(3) Suppose A intersects every YL set in Γ. Show that A is rank-determining. You may wish to use the following strategy.
    - Let $q, \mathcal{O}$ be as above. Let a be in A, and let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by reversing a directed path from q to a. Show that $\mathcal{O}'$ contains a directed cycle, and hence there are at least two directed paths in $\mathcal{O}$ from q to a.
    - Let U be the set of points in Γ (including q itself) that can be reached from q by precisely one directed path in $\mathcal{O}$. Show that U is connected.
    - Let X be a connected component of the complement of U. Since $\mathcal{O}$ restricted to X is still acyclic, it has a source v. Show that *outdeg*$_X(v)$ is at least 2.

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Conclude from this that U is a YL set disjoint from A. Therefore if A intersects every YL set, it must be rank-determining.

(4) Conclude the statement of the theorem and exhibit the necessity of the "loopless" condition. More precisely, exhibit a graph with model containing loops, such that the vertices of valence at least 3 do not form a rank determining set.

(5) Let T be a spanning tree for a graph Γ of genus g. Choose points $p_1, \ldots, p_g$ in the interior of every edge of $\Gamma \setminus T$, and a point p_0 of T. Show that $\{p_0, \ldots, p_g\}$ is a rank determining set[1].

Riemann–Roch for metric graphs.

(1) Sketch a proof of the Riemann–Roch theorem for metric graphs by pointing out the required modifications in the proof for finite graphs.

(2) Let G be a finite graph, and let Γ be the graph obtained by giving each edge length 1. Show that for every divisor D, we have $r_G(D) = r_\Gamma(D)$.

(3) Conclude that Riemann–Roch for finite graphs can be deduced directly from Riemann–Roch for metric graphs.

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