PROBLEM SET 5 - VALUATION THEORY AND BERKOVICH CURVES

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Let K be a field and $(\Gamma, +, \geq)$ a totally ordered abelian group. A *valuation* on K is a map

$$\nu : K^{\times} \to \Gamma,$$

such that

- $\nu(ab) = \nu(a) + \nu(b)$.
- $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

By convention, we set $\nu(0) = \infty$ and extend the order on Γ to $\Gamma \cup \{\infty\}$ by setting $\infty \geq \gamma$ for all $\gamma \in \Gamma$.

• The *valuation ring associated to* ν is the set of elements in K having nonnegative valuation.

• Alternatively, a subring $B \subset K$ is said to be a *valuation ring* if for every $x \in K^{\times}$ at least one of the set $\{x, x^{-1}\}$ lies in B.

Reminders on Valuations and Valuation Rings.

(1) Let K be a field and $\nu : K^{\times} \to \Gamma_1$ and $\mu : K^{\times} \to \Gamma_2$ be valuations. Define a relation $\nu \sim \mu$ if there is an order-preserving isomorphism of groups $\Phi : \Gamma_1 \to \Gamma_2$ such that for all $z \in K^{\times}$

$$\mu(z) = \Phi(\nu(z)).$$

Observe that this is an equivalence relation.

(2) Prove that the valuations on a field K up to the equivalence above are in natural bijection with the valuation subrings of K.

(3) Show that the only valuation on a finite field is trivial.

(4) Let B be a valuation ring. Prove that B is local with maximal ideal

$$\mathfrak{m}_B = \{z \in B : \nu(z) > 0\}.$$

(5) Prove that B is integrally closed.

(6) Show that B is noetherian if and only if $\nu(K^{\times})$ is isomorphic to $\mathbb{Z}$ or $\{0\}$. Deduce that an algebraically closed nontrivially valued field must have non-Noetherian valuation ring.

Date: July 29, 2016.

Field Extensions. Henceforth, assume that Γ has real rank 1, i.e. it is a subgroup of $\mathbb{R}$ with the induced order. Let $|\cdot|$ be the absolute value on K induced by ν. Namely, choose $\alpha \in \mathbb{R}$ with $1 < \alpha$ and set

$$|z| = \alpha^{-\nu(z)} \in \mathbb{R}_{\geq 0}.$$

We say that the valued field K is *complete* with respect to ν if the metric space $(K, |\cdot|)$ is complete.

A *non-archimedean field* is a field that is complete with respect to a real rank 1 valuation. We will denote the residue field $B/\mathfrak{m}_B$ by $\widetilde{K}$.

(1) Show that $\overline{\mathbb{Q}_p}$ (the algebraic closure of the p-adics) and the field Puiseux series are not complete.
(2) Suppose L is a finite extension of a complete discretely valued field K. Prove that the valuation extends uniquely to L. Deduce that valuation on K extends uniquely to the algebraic closure.

    Remark: Note that the discreteness hypothesis is not necessary, but substantially simplifies the proof. Given a field, complete with respect to a real rank-1 valuation, the valuation extends uniquely to algebraic extensions. For higher rank valuations, one requires a Henselian hypothesis on the valuation ring of the field. Complete rank-1 valuation rings are automatically Henselian. See VI §8 of "Algèbre commutative" by Bourbaki.

(3) Give an example of a valued field K and two distinct valuations on $K(t)$ restricting to the given valuation on K.
(4) Let L/K be a finite extension of non-archimedean fields. Let e be the index of the subgroup $\nu(K^\times)$ in $\nu(L^\times)$. Let f be the degree of the extension of residue fields $\widetilde{L}/\widetilde{K}$. Prove that

$$ef \leq [L : K].$$

    You may want to use the following strategy:
    - Choose coset representatives for the quotient $\nu(L^\times)/\nu(K^\times)$ and lifts $x_1, \ldots, x_e$ of these elements to $L^\times$.
    - Choose f elements $y_1, \ldots, y_f$ of the valuation ring of L whose reductions are linearly independent over $\widetilde{K}$.
    - We wish to prove that the elements $x_i y_j$ are linearly independent over K. Assume there is a relation

    $$\sum a_{ij} x_i y_j = 0,$$

    and consider the monomial term, indexed by say (i_0, j_0), of minimal valuation. Consider another monomial term (i_1, j_1) of minimal valuation and inspect the index i_1. How does it relate to i_0?
    - Deduce a linear dependence between the y_j's.

(5) **(Abhyankar's Inequality)** Suppose L/K is a finitely generated extension of non-archimedean fields. Let s be the rank of the $\mathbb{Q}$-vector space

$$[\nu(L^\times)/\nu(K^\times)] \otimes_\mathbb{Z} \mathbb{Q},$$

and let t be the transcendence degree of the extension of residue field $\widetilde{L}/\widetilde{K}$. Show that $s + t \leq \text{tr.deg}(L/K)$.

You will find a similar strategy as the previous question will work in the transcendental case. Rather than choosing coset representatives, choose basis elements in $[\nu(L^\times)/\nu(K^\times)] \otimes_\mathbb{Z} \mathbb{Q}$ and lift them to $L^\times$. Similarly, choose algebraically independent elements in L over K. Given a polynomial relation between these terms, inspect the monomial terms of minimal valuation and use the argument above.

Points on Berkovich Curves. Let A be a K-algebra of finite type and $X = Spec(A)$ the associated affine scheme. The *Berkovich analytification* of X is the set of *ring valuations*

$$X^{an} = \{\text{val} : A \to \mathbb{R} \sqcup \{\infty\} : \text{val}|_K = \nu\}.$$

Recall that ring valuations are defined just as valuations on fields, with the exception that nonzero elements can take the value ∞.

(1) Prove that the set of closed points of X embeds naturally into X^{an}.
(2) Given a point $x = \text{val}_x$ in X^{an}, prove that $\mathfrak{p}_x = \text{val}^{-1}(\infty)$ is a prime ideal in A.

Given $x \in X^{an}$ define the analytic residue field $H(x)$ as the completion with respect to val_x of the fraction field of $A/\mathfrak{p}_x$. The double residue field at x is the residue field of $H(x)$. We define $s(x)$ and $t(x)$ to be the parameters in Abhynakar's inequality with respect to the extension $H(x)/K$. From now on let $A = K[t]$ and $X^{an} = \mathbb{A}^1_{an}$.

(3) Let x be a point corresponding to a disk of radius 0 ("Type I"). Show that $s = t = 0$.
(4) Let x be a point corresponding to a disk of k-rational radius ("Type II"). Show that $\nu(H(x))^\times = \nu(K^\times)$, and that the double residue field equals $k((t))$, where k is the residue field of K. Conclude that $s(x) = 0$ and $t(x) = 1$.
(5) Let x be a point corresponding to a disk of non k-rational radius ("Type III"). Show that $\nu(H(X)^\times)$ properly contains $\nu(K^\times)$. Let $f = g/h$ be a rational function on the line with $\text{val}_x(f) = 0$. Show that the valuation of f is determined by a single monomial in each of the polynomials g, h and consequently, the reduction of f is in k. Conclude that $s(x) = 1$ and $t(x) = 0$.
(6) Show that for any other point of $\mathbb{A}^1_{an}$, $s(x) = t(x) = 0$ ("Type IV").