PROBLEM SET 6 - LINEAR SYSTEMS ON METRIC GRAPHS

DHRUV AND YOAV

In what follows, we discuss the polyhedral structure on the linear system of a divisor on a graph. We work in the metric setting and let Γ be a metric graph with real valued edge lengths. Denote by $(\mathbb{R}, \oplus, \odot)$ the min-plus semiring on the set $\mathbb{R}$.

Let D be a divisor on Γ. Define the set

$$R(D) = \{f : D + (f) \text{ is effective}\}.$$

Here, f is a piecewise linear function on Γ and (f) is the associated principal divisor. We view this set as a subset of the set of real valued functions on Γ.

(1) Prove that $R(D)$ is closed under tropical addition and the addition of globally constant functions. In other words, prove that $R(D)$ is a tropical semimodule.

An element $f \in R(D)$ is said to be *extremal for* D or simply *extremal*, if for any $g, g' \in R(D)$ such that $g \oplus g' = f$, then either $f = g$ or $f = g'$.

(2) Prove that f is extremal if there exist no two proper subgraphs of Γ, whose union is Γ, such that each subgraph can be fired on the divisor $D + (f)$ and the result be effective.

A cut set is a collection of points on Γ whose deletion leaves Γ disconnected. *Smooth* cut sets are those consisting only of divalent vertices.

(3) Let f be extremal. Prove that the support of $D + (f)$ does not contain a smooth cut set.
(4) Prove that the slopes of functions $f \in R(D)$ are bounded by a constant depending only on D. Deduce that the possible values of such f on the vertices of Γ is finite up to additive scaling.
(5) Using the above exercise, show that the set S, of functions in $R(D)$ such that the support of $D + (f)$ contains no smooth cut set, is a finite set.

(6) Prove that S generates $R(D)$ as a tropical semimodule.

Let $\mathbf{1}$ denote the constant function on Γ, equal to 1 at every point. Observe that there is a natural bijection between the sets $R(D)/\mathbf{1}$ and the set

$$|D| = \{D + (f) : f \in R(D)\}.$$

We will use this identification implicitly throughout the remainder of this worksheet.

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(7) Let Γ be a graph with a chosen finite graph model. Convince yourself that $Sym^d(\Gamma)$ admits a polytopal cell decomposition. How does this decomposition vary when varying the model on Γ.

Identify each open edge with an interval $(0, \ell(e))$, where $\ell(e)$ is the length of e. Note that this orients the graph. Convince yourself that $Sym^k(e)$ is naturally identified with an open simplex in $\mathbb{R}^k$.

(8) A cell of $|D|$ is indexed by the following data:
- An integer d_v for each vertex $v \in \Gamma$.
- An integer d_e and an ordered decomposition $d_e^{(1)} + \cdots + d_e^{(r_e)}$ for each edge e.
- An integer m_e for each edge of Γ.

A divisor D' belongs to such a cell if
- For each vertex v, $D'(v) = d_v$.
- Upon restriction to e, D' is given by $\sum_i d_e^i x_i$, where $0 < x_1 < \cdots < x_{r_e} < \ell(e)$.
- The slope of f on the beginning of e is m_e, where f is the function such that $D' = D + (f)$.

If $D' \sim D$, prove that $|D|$ and $|D'|$ are canonically identified as cell complexes.

(9) Compute the cell structure of any divisor D on a tree, and for the divisor $D = 3p$ for a point p on a circle Γ.

(10) For a divisor D' in $|D|$, let $I_{D'}$ be the set of points of the support of D in the interior of edges. Show that the dimension of the cell of $|D|$ containing D in its interior equals one less than the number of connected components of $\Gamma \setminus I_D$.

(11) Let V be the set of zero cells of $|D|$ and $S(V)$ the set of functions in $R(D)$ such that $D + (f)$ lies in V. Show that any element of S, as defined in Exercise (5) , is a cell in $|D|$ of dimension 0. Deduce that $S(V)$ contains S and hence generates $R(D)$.

(12) Show that every closed cell in $|D|$ is closed under tropical addition.

(13) Show that every closed cell in $|D|$ is generated by its vertices.

(14) Compute the cell structure of the canonical divisor on the dumbbell graph.