PROBLEM SET 7 - MATRIX-TREE THEOREM

DHRUV AND YOAV

Let $\{v_1, \dots, v_n\}$ and $\{e_1, \dots, e_m\}$ be the set of vertices and edges of a finite graph G. The *Laplacian* matrix Q of G is the $n \times n$ matrix whose (i,j)-entry is minus the number of edges between v_i and v_j for $i \neq j$, and is the valency of i when $i = j$. Given an orientation on G, the oriented adjacency matrix A is the $n \times m$ matrix whose (i,j)-entry is given by

$$A(i,j) = \begin{cases} 1 & \text{the edge } e_j \text{ is oriented towards } v_i. \\ -1 & \text{the edge } e_j \text{ is oriented away from } v_i. \\ 0 & \text{otherwise} \end{cases}$$

(1) Show that the image of the map

$$\Delta : \mathbb{Z}^{|V(G)|} \to \mathrm{Div}(G)$$
$$\Delta(v) = Q \cdot v$$

coincides with the set of principal divisors.

(2) Fix an orientation on G. Let Q' be the matrix Q with the first row and column deleted, and let A' be the matrix A with the first row deleted. Show that $Q = AA^t$ and $Q' = (A')(A')^t$.

(3) Use the Cauchy-Binet formula to show that $\det(Q') = \sum_S \det(A'_S)^2$, where S ranges over all subsets of $E(G)$ of size $n-1$. (Bonus: How is the Cauchy-Binet formula a generalization of the Pythagorean theorem?)

(4) Show that $\det(A'_S) = \pm 1$ if S corresponds to a spanning tree of G, and $\det(A'_S) = 0$ otherwise. Conclude that $\det(Q')$ is equal to the number of spanning trees in G (this is Kirchoff's theorem).

(5) Find a bijection between the Jacobian of G and the cokernel of the map $\Delta' : \mathbb{Z}^{|V(G)-1|} \to \mathbb{Z}^{g-1}$ given by $\Delta'(v) = Q' \cdot v$.

(6) Use the theory of the Smith Normal Form to show that the determinant of Q' is equal to $|\mathrm{Jac}(G)|$, and therefore $|\mathrm{Jac}(G)|$ is the number of spanning trees in G.

(7) Determine the structure of $\mathrm{Jac}(K_n)$. Use this to give a refinement of the classical Cayley formula that the number of spanning trees in K_n is n^{n-2}.

(8) We will now construct a bijection between $\mathrm{Jac}(G)$ and spanning trees of G which will, in particular, give another proof of Kirchhoff's theorem not making use of the Cauchy-Binet formula.

    (a) Fix an ordering of the edges of G and a vertex q. Given a q-reduced divisor D of degree 0, run Dhar's burning algorithm on D but with a "controlled burn": any time you have a choice for which edge to burn next, choose the one which is largest in the fixed ordering of $E(G)$. Any time a vertex is burned through (because the firefighters are overwhelmed), mark the edge

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which burned through it. When the burning algorithm is complete, show that the set of marked edges is a spanning tree of G.

(b) Conversely, given a spanning tree T, run a modified version of Dhar's algorithm where you start burning from q and burn through a vertex v as soon as you use an edge in T. When this happens, set

$$D(v) = (\# \text{ of burnt neighbors of } v) - 1.$$

Show that D is q-reduced and that the procedures in (a) and (b) are inverse to one another.