TROPICAL SCHEME THEORY

Idempotent semirings

Definition 0.1. *A semiring is* $(R, +, \cdot, 0, 1)$ *such that* $(R, +, 0)$ *is a commutative monoid (so* $+$ *is a commutative associative binary operation on* R *and* 0 *is an additive identity),* $(R, \cdot, 1)$ *is a monoid (so* $\cdot$ *is an associative binary operation on* R *and* 1 *is a two-sided identity for* $\cdot$*) satisfying*
distributivity: $a \cdot (b + c) = (ab) + (ac)$ *and* $(a + b)c = (ac) + (bc)$ *and*
"with zero": $0 \cdot a = 0 = a \cdot 0$ *for all* $a \in R$*.*

Example 0.2. The semiring of non-negative real matrices $M_{n\times n}(\mathbb{R}_{\geq 0})$.

Definition 0.3. *A semiring is commutative if* $ab = ba$ *for all* $a, b \in R$*.*

All semirings will be commutative unless otherwise specified.

Example 0.4. $\mathbb{R}_{\geq 0}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n], \mathbb{Z}_{\geq 0}[x_1^{\pm}, \ldots, x_n^{\pm}], \mathbb{R}_{\geq 0} \cup \{\infty\}$ (note: $0 \cdot \infty = \infty \cdot 0 = 0$ to satisfy axioms), $\mathbb{R}[x]$.

Definition 0.5. *A semiring is additively idempotent is* $a + a = a$ *for all* a*.*

Idempotent semiring will mean commutative and additively idempotent semiring unless otherwise specified.

Note that the literature also contains extensive discussion of multiplicatively idempotent semirings, such as distributive lattices. These will be less relevant for our purposes.

Example 0.6. The tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$.

: The additive and multiplicative identities of an idempotent semiring may not be denoted by 0 and 1, and a semiring may contain other elements named 0 and 1. When necessary, we will specify the additive and multiplicative identities of a semiring R by 0_R and 1_R, respectively. For instance, $0_{\mathbb{T}} = \infty$ and $1_{\mathbb{T}} = 0$.

Example 0.7. The laciport[1] numbers $\mathbb{L}$=$(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$.

Example 0.8. Let A be a commutative ring. The set R of ideals in A is an idempotent semiring with $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$, $0_R = (0)$, and $1_R = (1) = A$.

Question 0.9. *What properties of the ring* A *are reflected in the semiring of its ideals?*

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[1]Named after Max Laciport, a mid-century algebraist and close personal friend of the Basque mathematician Hirune Mendebaldeko.

Comment 0.10 (Dave)**.** If you consider this example in the case where A is a valuation ring with value group Γ, then you get $\Gamma_{\geq 0} \cup \{\infty\}$, with addition being min and multiplication being $+$. In particular, for a discrete valuation ring, this gives $\mathbb{N} \cup \{\infty\}$ with tropical structure.

Comment 0.11 (Nati)**.** There is another idempotent semiring structure on the set of ideals in A, with multiplication given by intersection. This reflects the (distributive) lattice structure on the set of ideals.

Example 0.12. If R is an idempotent semiring and X is a set then R^X is an idempotent semiring.

Definition 0.13. *A topological semiring is a semiring R with a topology on R such that $+, \cdot$ are continuous.*

Example 0.14. The semiring of tropical numbers $\mathbb{T}$ carries a natural topology in which the map $-\log : \mathbb{R}_{\geq 0} \to \mathbb{T}$ is a homeomorphism. Then $\mathbb{T}_\Gamma = \Gamma \cup \{\infty\}$ carries the subspace topology for any subgroup $\Gamma \subset \mathbb{R}$. More generally, the order topology induces a topological semiring structure on $\Gamma \cup \infty$ for *any* totally ordered abelian group Γ.

Example 0.15. If R is a topological idempotent semiring and X is a topological space then $\{f : X \to R \mid f \text{ is continuous}\}$ is an idempotent semiring.

Example 0.16. Let $\Gamma \subset \mathbb{R}$ be an additive subgroup. Then $\mathrm{CPL}_{\mathbb{Z},\Gamma}(\mathbb{R}) \subset \{\text{continuous} f : \mathbb{R} \to \mathbb{T}\}$ is the subsemiring generated by the linear functions $t \mapsto at + b$ with $a \in \mathbb{Z}$ and $b \in \Gamma$.

These functions are continuous, convex, and PL with respect to a finite decomposition of $\mathbb{R}$ into closed (possibly unbounded) intervals (plus the constant function ∞). One may also consider the semiring of continuous, convex functions that are $\mathrm{PL}_{\mathbb{Z},\Gamma}$ with respect to a locally finite decomposition. Roughly speaking, functions that are CPL with respect to finite decompositions come from tropicalizing algebraic functions, while locally finite decompositions arise when tropicalizing analytic functions.

Example 0.17. The previous example generalizes to CPL functions on higher dimensional spaces, such as the semiring $\mathrm{CPL}_{\mathbb{Z},\Gamma}(\mathbb{R}^n)$ generated by $v \mapsto \langle u, v\rangle + b$ for $u \in \mathbb{Z}^n$ and $b \in \Gamma$.

For Δ the support of some polyhedral complex in $\mathbb{R}^n$, we may also consider $\mathrm{CPL}_{\mathbb{Z},\Gamma}(\Delta)$, the semiring of such functions restricted to Δ.

Example 0.18. When we consider tropicalizing rational functions, we drop the convexity requirement. This gives us $\mathrm{PL}_{\mathbb{Z},\Gamma}(\mathbb{R})$ (or $\mathbb{R}^n$, Δ) generated by $\{(f - g) \mid f, g \in \mathrm{CPL}_{\mathbb{Z},\Gamma}\}$.

Example 0.19. Let Y be a toric variety. Then we have $\mathrm{Trop}(Y)$, which contains the vector space $N_{\mathbb{R}} \cong \mathbb{R}^n$ as an open and dense subset. The set of continuous functions $\mathrm{Trop}(Y) \to \mathbb{T}$ whose restriction to $N_{\mathbb{R}} \cong \mathbb{R}^n$ is in $\mathrm{CPL}_{\mathbb{Z},\Gamma}$ forms an idempotent semiring.

Exercise 0.20. Compute this semiring when $Y = U_\sigma$ is affine. Because continuous functions to a Hausdorff spaces are determined on dense sets, this is a subsemiring of $\mathrm{CPL}_{\mathbb{Z},\Gamma}(N_\mathbb{R})$. Which of these CPL functions extend to $\mathrm{Trop}(U_\sigma)$?

Example 0.21. The Boolean semiring $\mathbb{B} = \{0, 1\}$.

Question 0.22 (Open Problem(?))**.** *Classify the finite (commutative) idempotent semirings.*

Comment 0.23 (Kalina)**.** The only idempotent division-semiring is $\mathbb{B}$.

Definition 0.24. *A semiring R is algebraically closed if for each $a \in R$ and each positive integer n, there is some $b \in R$ such that $b^n = a$.*

Example 0.25. Algebraically closed semirings include $\mathbb{T}$, $\mathbb{T}_\mathbb{Q}$, and more generally $\mathbb{T}_\Gamma$ for $\Gamma \subset \mathbb{R}$ divisible. But $\mathrm{CPL}_{\mathbb{Z},\Gamma}(\mathbb{R})$ is not, because if you divide by $n \geq 2$ you get fractional slopes.

This definition seems to be standard in the literature. Is it useful for our purposes? One natural seeming alternative would be to require some analogue of existence of solutions to all nonconstant polynomials.

Definition 0.26. *An ideal $\mathfrak{a} \subset R$ in a semiring R is a subset that is closed under addition and under multiplication by R. That is*

$$\mathfrak{a} + \mathfrak{a} \subset \mathfrak{a}$$

and

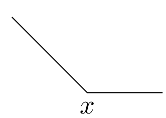
$$R \cdot \mathfrak{a} \subset \mathfrak{a}.$$

Example 0.27. Any subset of R generates an ideal: for $S \subset R$ we have $(S) = \{a_1 s_1 + \cdots + a_r s_r \,|\, a_i \in R, s_i \in S\}$. This is the smallest ideal containing S.

Example 0.28. Any intersection of ideals is an ideal.

Example 0.29. (Corresponding to the fact that in commutative algebra, we think of an ideal as functions vanishing on some set.) Given R, X and $S \subset X$ we have that $\{f : X \to R \,|\, f(S) = 0_R\}$ is an ideal of R^X. Similarly if X is a topological space or a tropicalized toric variety, we get ideals in the corresponding semirings of functions.

Example 0.30. $\{f \in \mathrm{CPL}_\mathbb{Z}(\mathbb{R}) \,|\, f \text{ bends at } x \in R\}$ is an ideal. We say that "f bends at x" if f is not linear at x. This uses convexity: If we don't require convexity then we get that the minimum of



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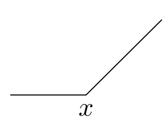

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both of which bend at x, does not bend at x.

Example 0.31. $\{f \in \mathrm{CPL}(\mathbb{A}^1_{\mathrm{trop}}) \mid f \text{ is decreasing towards } -\infty\}$. These correspond to functions that have a pole at ∞, i.e. polynomials of positive degree.

Neither of these last two examples is the tropicalization of an ideal of polynomials. But Example 1.30 looks more like an ideal. That Example 1.31 is an ideal seems to be a relic of the fact that in semirings we don't have cancellation. Maybe we don't fully understand how ideals in semirings relate to ideals in rings. We are also missing a satisfactory understanding of what the analogue of prime ideals (or irreducible varieties in tropical scheme theory) should be.

Definition 0.32. *A homomorphism of semirings $f : R \to R'$ is a map such that*

$$\begin{aligned} f(a+b) &= f(a) + f(b), \\ f(a \cdot b) &= f(a) \cdot f(b), \\ f(0_R) &= 0_{R'}, \text{ and} \\ f(1_R) &= 1_{R'}. \end{aligned}$$

Example 0.33. The natural map $\mathbb{T}[x_1, \ldots, x_n] \to \mathrm{CPL}_{\mathbb{Z}}(\mathbb{R}^n)$ sending $\mathbb{T}$ to constant functions and x_i to the i^{th} coordinate function, is a homomorphism. Note that this map takes monomials to affine linear functions:

$$cx_1^{b_1} \cdots x_n^{b_n} \mapsto \langle (b_1, \ldots, b_n), \cdot \rangle + c.$$

It is not injective; a term may not be relevant if it never attains the minimum. This is very important in the work of Giansiracusa-Giansiracusa which we will be studying in the next few weeks.

If K is a valued field, then the natural tropicalization maps

$$K[x] \to \mathbb{T}[x]$$

and

$$K[x^{\pm}] \to \mathrm{CPL}_{\mathbb{Z}}(\mathbb{R})$$

are <u>not</u> homomorphisms. This is because $\mathrm{val}(a+b) \neq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ in general.

Example 0.34. The preimage of an ideal under a homomorphism is an ideal. This gives another reason why the tropicalization maps are not homomorphisms, as the inverse image of the bend ideal is not an ideal.

Proposition 0.35. *Every ideal of R is contained in a maximal ideal.*

The proof is the usual Zorn's Lemma argument from commutative algebra.

References for people interested in learning more about the general theory of semirings include [Go99] and [HW98]. Also, see pages 16-17 in [Go99] for many further references regarding applications of $\mathbb{T}$ in other areas (especially to optimization and theoretical CS).

References

[Go99] J. Golan, *Semirings and Their Applications*, Kluwer, Dordrecht, 1999

[HW98] U. Hebisch and H. Weinert, *Semirings: algebraic theory and applications in computer science*, Series in Algebra, vol. 5, World Scientific Publishing Co., Inc., River Edge, NJ, 1998, Translated from the 1993 German original