TROPICAL SCHEME THEORY

10. Tropical ideals

The idea of tropical scheme theory in general is the following. An ideal $I \subset K[x_1, \dots, x_n]$ gives rise to a variety $V(I) \subset \mathbb{A}^n_K$ a variety. With classical tropicalization we get a polyhedral complex $\operatorname{trop}(V) \subset \bar{\mathbb{R}}^n$, where $\bar{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus = \min, \odot = +)$. Scheme theoretic tropicalization remembers more information. For example, the data of $\mathcal{B}(I)$, the bend relations, are equivalent to remembering $\operatorname{trop}(I) = \langle \operatorname{trop}(f) \mid f \in I \rangle$ (you need "ideal generated by" if the valuation isn't surjective). So we get some ideals in $\bar{\mathbb{R}}[x_1, \dots, x_n]$, but which ideals in $\bar{\mathbb{R}}[x_1, \dots, x_n]$ do we like for tropical geometry?

It seems we don't want to consider all of the ideals.

<u>Some complications</u>:

(1) $\bar{\mathbb{R}}[x_1, \dots, x_n]$ is not Noetherian. For example,

$$\operatorname{trop}(\langle x - y \rangle) = \left\langle x \oplus y, x^2 \oplus y^2, x^2 \oplus y^2, \dots \right\rangle$$

is not finitely generated, and so the chain of ideals given by taking the first n of these gives an infinite ascending chain of ideals in $\bar{R}[x, y]$.

(2) If $I \subset \bar{\mathbb{R}}[x_1, \dots, x_n]$ we define $V(I) := \{x \in \bar{\mathbb{R}}^n \mid$ the minimum in $f(x)$ is achieved twice $\forall f \in I\}$. This gives too much.

Proposition 10.1. *Any convex closed subset S of $\mathbb{R}^n$ whose affine hull has dimension at most $n-1$ (i.e. which is contained in some affine hyperplane) and such that $\operatorname{aff}(S)$ is a rational subspace is of the form $V(I)$ for some ideal I.*

Proof. Note that any such S is an intersection of rational hyperplanes and rational half-hyperplanes. Any rational hyperplane is the bend locus of a binomial. And we can get any rational half-hyperplane by intersecting two tropical varieties which are shifts of each other. $\square$

So we restrict which ideals we look at.

Definition 10.2. *An ideal $I \subset \bar{\mathbb{R}}[x_1, \dots, x_n]$ is a <u>tropical ideal</u> if*

$$I_d := \{f \in I \mid \deg f \leq d\} \subset \bar{\mathbb{R}}^{\mathrm{Mons}_{\leq d}}$$

(where $\mathrm{Mons}_{\leq d}$ is the set of monomials of degree $\leq d$) is a tropical linear space for every $d \geq 0$.

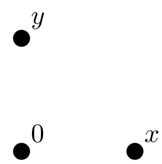
Equivalently, I is a tropical linear space if for any $f, g \in I$ with some monomial $\underline{x^{u}}$ of degree at most d such that $[f]_{x^u} = [g]_{x^u}$ then there is some $h \in I$ such that $[h]_{x^u} = \infty$ (tropical 0) and for any monomial x^v, $[h]_{x^v} \geq \min([f]_{x^v}, [g]_{x^v})$ with equality if $[f]_{x^v} \neq [g]_{x^v}$.

Date: October 26, 2017, *Speaker*: Felipe Rincón, *Scribe*: Netanel Friedenberg.

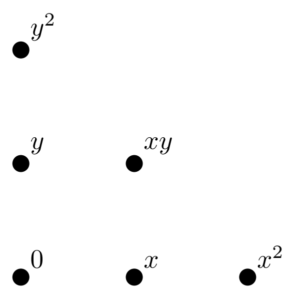
This is a proposed definition to solve some of the above problems.

Example 10.3.

- If $I = \mathrm{trop}(J)$ for $J \subset K[x_1, \ldots, x_n]$ then J is a tropical ideal. In this case we say that I is realizable.
- $I = \langle x \oplus y \rangle$ is not a tropical ideal: In degree 2 we have $x^2 \oplus xy$ and $xy \oplus y^2$, but this would force $x^2 \oplus y^2 \in I$.
  Warning: Tropical ideals are generally not finitely generated. (There are trivial exceptions, such as monomial ideals.)
- There are non-realizable tropical ideals. Take $I \subset \bar{\mathbb{R}}[x, y]$ generated (as a semi-module) by polynomials of the form $f = \bigoplus_{x^u \in C} x^u$ for C a minimal collection of $k + 1$ monomials in $\mathrm{Mons}_{\leq d}$ inside a "standard" triangle of size k. For example, in degree ≤ 1 we have

  which is a standard triangle of size 2, so $x \oplus y \oplus 0 \in I$. In degree ≤ 2 we have

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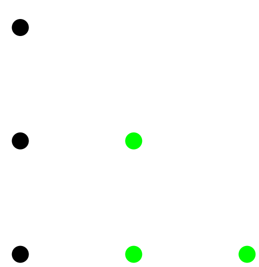
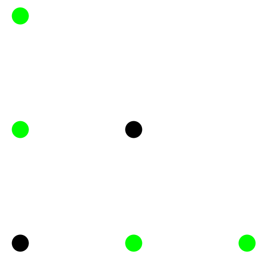
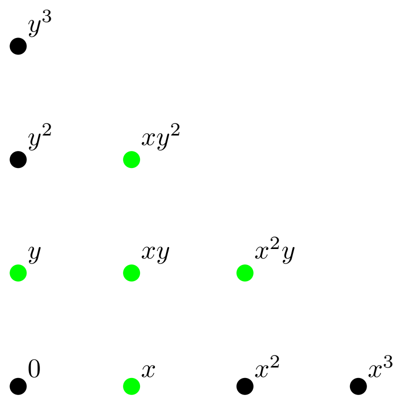
  

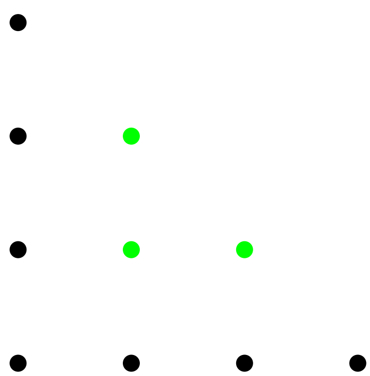
  we see that $x \oplus x^2 \oplus xy \in I$, and from



we see that $x \oplus x^2 \oplus y \oplus y^2 \in I$. Note that in degree ≤ 3



$x \oplus y \oplus xy \oplus x^2y \oplus xy^2$ is <u>not</u> in I because we already have



$xy \oplus x^2y \oplus xy^2$ in I. It is not hard to see that this gives a tropical ideal. (What we have done is described the circuits of the matroid.) $V(I) = V(x \oplus y \oplus 0)$. But I is not realizable: If $I = \mathrm{trop}(J)$ then say $x + y + 1 \in J$ so $(x + y + 1)(x^2 + y^2 + 1 - xy - x - y) = x^3 + y^3 + 1 - 3xy \in J$ but $\mathrm{trop}(x^3 + y^3 + 1 - 3xy)$ is not in I because it has too few monomials. (Note that up to scaling each of the variables by a scalar with valuation 0, which doesn't change the tropicalizations, we will will have $x + y + 1 \in J$, so this is the only case we need to consider.)

<u>Nice things about tropical ideals:</u>

If I is a tropical ideal then we get a Hilbert function $H_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ $H_I(d) = \dim(I_d)$ - while dimension is not well-behaved for arbitrary semi-modules, it is well-behaved for tropical linear spaces. If $I = \mathrm{trop}(J)$ then $H_I = H_J$.

Proposition 10.4. *$H_I(d)$ is eventually polynomial for any tropical ideal.*

Theorem 10.5. *There is no infinite ascending chain*

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdot$$

of $\underline{\text{tropical}}$ ideals.

Theorem 10.6. *If I is a tropical ideal then $V(I)$ is (the support of) a $\underline{\text{finite}}$ polyhedral complex.*

Let I be a tropical ideal. We have the bend congruence

$$\mathcal{B}(I) = \langle f \sim f_{\hat{u}} \mid f \in I, u \in \mathrm{supp}(f) \rangle .$$

Then we can recover I as $I = \{f \in \bar{\mathbb{R}}[x_1, \dots, x_n] \mid \mathcal{B}(f) \subset \mathcal{B}(I)\}$. The main point is that the proof of this for the classical case (i.e. tropicalized ideals) really only took advantage of duality for valuated matroids.

Let's talk about why the Hilbert function is eventually polynomial and why there are no infinite ascending chains of tropical ideals. They use the ideal of initial degenerations.

Consider a weight vector $w \in \mathbb{R}^n$ (think of as assigning a weight to each variable). For $f \in \bar{\mathbb{R}}[x_1, \dots, x_n]$ write $f = \bigoplus_{u \in \mathbb{N}^n} a_u \odot x^u$, then

$$\mathrm{in}_w f := \bigoplus_{a_u \odot w^u = f(w)} x^u \in \mathbb{B}[x_1, \dots, x_n].$$

If $I \subset \bar{\mathbb{R}}[x_1, \dots, x_n]$ is an ideal then $\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \mid f \in I \rangle$.

$\underline{\text{Key fact}}$: If I is a tropical ideal then $\mathrm{in}_w(I)$ is also a tropical ideal. The bases of $M_d(\mathrm{in}_w(I))$ are the bases B of $M(I_d)$ such that $p_d(B) - \sum_{i \in B} w_i$ is minimal. In the trivially valued case this is taking the corresponding face of the matroid polytope. In the non-trivially valued case this goes by taking the correct piece of the corresponding regular subdivision of the matroid polytope.

In particular, because the bases don't change size, this gives us that $H_{\mathrm{in}_w(I)} = H_I$.

Proof that $H_I(d)$ is eventually polynomial. If I is a tropical ideal then for generic $w \in \mathbb{R}^n$ $\mathrm{in}_w(I)$ is a monomial ideal. Because "monomial ideals are the same classically and tropically", $H_{\mathrm{in}_w(I)}$ is eventually polynomial, $H_I = H_{\mathrm{in}_w I}$ is eventually polynomial. □

Proof that tropical ideals satisfy the ACC. Take an infinite chain

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

of tropical ideals. For very generic $w \in \mathbb{R}^n$ (we just have to avoid a countable collection of codimension-1 things)

$$\mathrm{in}_w(I_1) \subset \mathrm{in}_w(I_2) \subset \mathrm{in}_w(I_3) \subset \cdots$$

is a chain of monomial ideals. But chains of monomial ideals must stabilize. So $H_{I_n} = H_{\mathrm{in}_w(I_n)}$ must stabilize. But if $I \subset J$ are tropical ideals with the same Hilbert function then we must have $I = J$. This follows from a basic matroidal fact–if $L_1 \subset L_2$ are tropical linear spaces of the same dimension then $L_1 = L_2$. $\square$