TROPICAL SCHEME THEORY

2. Commutative algebra over idempotent semirings

Proposition 2.1 (Noetherian property for ideals in semirings)**.** *The following are equivalent:*

(1) *R is Noetherian (in the sense that it satisfies the ascending chain condition).*
(2) *Any nonempty collection of ideals of I has a maximal element.*
(3) *Every ideal of R is finitely generated.*

Proof. (1) $\Longrightarrow$ (2): Start with a collection $\mathcal{C}$ as in (2). Pick $I_1 \in \mathcal{C}$. If $I_1 \subsetneqq J$ for all $J \in \mathcal{C}$, we are done. Otherwise, pick $I_2 \in \mathcal{C}$ with $I_1 \subset I_2$. Iterating this, (1) tells us that the chain stabilizes, and it must stabilize at a maximal element of $\mathcal{C}$.
(2) $\Longrightarrow$ (3): Given I, let $\mathcal{C}$ be the collection of finitely generated ideals contained in I. By (2) there is some maximal element $H \in \mathcal{C}$. Write $H = \langle a_1, \ldots, a_n \rangle$. For each $b \in I$ let $H_b = \langle a_1, \ldots, a_n, b \rangle$. Then by maximality of H we have $H_b = H$, so in particular $b \in H$. Thus $I = H$ is finitely generated.
(3) $\Longrightarrow$ (1): If $(I_n)_{n\in\mathbb{N}}$ is an increasing chain of ideals, then letting $I = \bigcup_{n=1}^{\infty} I_n$ we have that I is finitely generated. Since the generators of I were already all in some I_n, we have that the chain stabilizes by I_n. □

A much more general statement is true. Note that (1) and (2) are equivalent in any partially ordered set P. (3) is equivalent to (1) and (2) whenever P consists of subsets of some set A ordered by inclusion, and there is a notion of "poset element generated by a subset" given by a map $2^S \to P$, $G \mapsto \langle G \rangle$ such that

- for all $G \subset S$, $G \subset \langle G \rangle$,
- for all $G \subset H \subset S$, $\langle G \rangle \subset \langle H \rangle$, and
- for all $I \in P$, $\langle I \rangle = I$

(that is, $G \mapsto \langle G \rangle$ is a "closure operator".)

Definition 2.2. *An ideal is maximal if it is not properly contained in any other (proper) ideal.*

Fact: Every (proper) ideal is contained in a maximal ideal.

Example 2.3. $\mathbb{B}[x]$ has a unique maximal ideal, consisting of all polynomials that are not just 1.

Definition 2.4. *An ideal $I \subset R$ is a prime ideal if whenever $HK \subset I$ for H, K ideals of R then $H \subset I$ or $K \subset I$.*

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Proposition 2.5. *Let $I \subset R$ be an ideal. The following are equivalent:*

(1) *I is prime.*
(2) *If $a, b \in R$ are such that $(a)(b) \subset I$ then $a \in I$ or $b \in I$.*
(3) *If $ab \in I$ then $a \in I$ or $b \in I$.*

Example 2.6. In $\mathbb{B}[x]$:

- $I = (x)$ is a prime ideal.
- $(x+1)$ is <u>not</u> prime since
$$(1 + x + x^3)(1 + x^2 + x^3) = (1 + x)^6.$$
- $(x, x+1)$ is a proper ideal of $\mathbb{B}[x]$.

Aside: In $R = \mathbb{C}[x]$, we have $(x, x+1) = R$ because $1 = x + 1 - x \in (x, x+1)$. Note that $(0) \subset (x)$ is a chain is a chain which can't be increased at the top by any proper ideal.

Example 2.7. For $A \subset \mathbb{N}\backslash\{0\}$, let $I(A) = \langle x, \{1 + x^n | n \in A\}\rangle$.
<u>Claim</u>: $I(A)$ is prime iff $\mathbb{N}\backslash A$ is an ideal of $\mathbb{N}$.
In particular, if we let $A_n = \mathbb{N}\backslash\{2^n\}$ for all $n \geq 0$ then we have the infinite increasing chain
$$(x) \subset I(A_0) \subset I(A_1) \subset \cdots.$$
So $\mathbb{B}[x]$ is not Noetherian! This example is due to F. Alarcón and D. Anderson [AA94].

Proposition 2.8. *Any maximal ideal is prime.*

Proof. Let $I \subset R$ be a maximal ideal. Say we have $xy \in I$ and $x \notin I$. Let
$$\begin{aligned} J &:= I + Rx \\ &= \{i + rx | i \in I, r \in R\}. \end{aligned}$$
Note that $I \subsetneq J$ because $x \in J$ but $x \notin I$. So $J = R$. Thus $1 = i + ax$ for some $i \in I$ and $a \in R$. Then $y = 1 \cdot y = (i + ax)y = iy + axy \in I + I \subset I$. That is, $y \in I$. ☐

Proposition 2.9. *Every prime ideal contains a minimal prime ideal.*

Proof. We argue the same way as for ideals of rings, using Zorn's lemma. ☐

In rings we have that if I is a prime ideal then I is irreducible.

Definition 2.10. *An ideal I of a semiring R is <u>irreducible</u> if for $H, K \subset R$ ideals we have that*
$$I = H \cap K \implies I = H \text{ or } I = K.$$
An ideal I of R is <u>strongly irreducible</u> if for $H, K \subset R$ ideals we have that
$$I \supset H \cap K \implies I \supset H \text{ or } I \supset K.$$

Proposition 2.11. *Let $I \subset R$ be an ideal. Then I is prime if and only if*

(1) *I is strongly irreducible, and*
(2) *for any ideal $H \subset R$, $H^2 \subset I \implies H \subset I$.*

Proof. ($\Rightarrow$) (2) is immediate. For (1), let $M \cap K \subset I$. Then $MK \subset M \cap K \subset I$ so $M \subset I$ or $K \subset I$.
($\Leftarrow$) Say M, K are ideals such that $MK \subset I$. Then we have that $(M \cap K)^2 \subset MK \subset I$. By (2) it follows that $M \cap K \subset I$, so by (1) $M \subset I$ or $K \subset I$. Thus I is prime. □

Question 2.12. *(spoiler): Would this proposition be useful for finding the right definition of a prime tropical ideal?*

References

[AA94] F. Alarcón and D. Anderson, *Commutative semirings and their lattices of ideals*, Houston Journal of Mathematics, Volume 20, No. 4, 1994