2. Commutative algebra over idempotent semirings

**Proposition 2.1** (Noetherian property for ideals in semirings). The following are equivalent:

1. $R$ is Noetherian (in the sense that it satisfies the ascending chain condition).
2. Any nonempty collection of ideals of $I$ has a maximal element.
3. Every ideal of $R$ is finitely generated.

**Proof.** (1) $\Rightarrow$ (2): Start with a collection $C$ as in (2). Pick $I_1 \in C$. If $I_1 \not\subset J$ for all $J \in C$, we are done. Otherwise, pick $I_2 \in C$ with $I_1 \subset I_2$. Iterating this, (1) tells us that the chain stabilizes, and it must stabilize at a maximal element of $C$.

(2) $\Rightarrow$ (3): Given $I$, let $C$ be the collection of finitely generated ideals contained in $I$. By (2) there is some maximal element $H \in C$. Write $H = \langle a_1, \ldots, a_n \rangle$. For each $b \in I$ let $H_b = \langle a_1, \ldots, a_n, b \rangle$. Then by maximality of $H$ we have $H_b = H$, so in particular $b \in H$. Thus $I = H$ is finitely generated.

(3) $\Rightarrow$ (1): If $(I_n)_{n \in \mathbb{N}}$ is an increasing chain of ideals, then letting $I = \bigcup_{n=1}^{\infty} I_n$ we have that $I$ is finitely generated. Since the generators of $I$ were already all in some $I_n$, we have that the chain stabilizes by $I_n$. □

A much more general statement is true. Note that (1) and (2) are equivalent in any partially ordered set $P$. (3) is equivalent to (1) and (2) whenever $P$ consists of subsets of some set $A$ ordered by inclusion, and there is a notion of “poset element generated by a subset” given by a map $2^S \to P, \{G\} \mapsto \langle G \rangle$ such that

- for all $G \subset S, G \subset \langle G \rangle$,
- for all $G \subset H \subset S, \langle G \rangle \subset \langle H \rangle$, and
- for all $I \in P, \langle I \rangle = I$

(that is, $G \mapsto \langle G \rangle$ is a “closure operator”.)

**Definition 2.2.** An ideal is maximal if it is not properly contained in any other (proper) ideal.

Fact: Every (proper) ideal is contained in a maximal ideal.

**Example 2.3.** $\mathbb{B}[x]$ has a unique maximal ideal, consisting of all polynomials that are not just 1.

**Definition 2.4.** An ideal $I \subset R$ is a prime ideal if whenever $HK \subset I$ for $H, K$ ideals of $R$ then $H \subset I$ or $K \subset I$.  

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Proposition 2.5. Let $I \subset R$ be an ideal. The following are equivalent:

1. $I$ is prime.
2. If $a, b \in R$ are such that $(a)(b) \subset I$ then $a \in I$ or $b \in I$.
3. If $ab \in I$ then $a \in I$ or $b \in I$.

Example 2.6. In $\mathbb{B}[x]$:

- $I = (x)$ is a prime ideal.
- $(x + 1)$ is not prime since
  $$(1 + x + x^3)(1 + x^2 + x^3) = (1 + x)^6.$$
- $(x, x + 1)$ is a proper ideal of $\mathbb{B}[x]$.

Aside: In $R = \mathbb{C}[x]$, we have $(x, x + 1) = R$ because $1 = x + 1 - x \in (x, x + 1)$. Note that $(0) \subset (x)$ is a chain which can’t be increased at the top by any proper ideal.

Example 2.7. For $A \subset \mathbb{N}\setminus\{0\}$, let $I(A) = \langle x, \{1 + x^n|n \in A\} \rangle$.

Claim: $I(A)$ is prime iff $\mathbb{N}\setminus A$ is an ideal of $\mathbb{N}$.

In particular, if we let $A_n = \mathbb{N}\setminus\{2^n\}$ for all $n \geq 0$ then we have the infinite increasing chain

$$(x) \subset I(A_0) \subset I(A_1) \subset \cdots.$$

So $\mathbb{B}[x]$ is not Noetherian! This example is due to F. Alarcón and D. Anderson [AA94].

Proposition 2.8. Any maximal ideal is prime.

Proof. Let $I \subset R$ be a maximal ideal. Say we have $xy \in I$ and $x \notin I$. Let

$$J := I + Rx = \{i + rx | i \in I, r \in R\}.$$  

Note that $I \subseteq J$ because $x \in J$ but $x \notin I$. So $J = R$. Thus $1 = i + ax$ for some $i \in I$ and $a \in R$. Then $y = 1 \cdot y = (i + ax)y = iy + axy \in I + I \subset I$. That is, $y \in I$. \hfill $\square$

Proposition 2.9. Every prime ideal contains a minimal prime ideal.

Proof. We argue the same way as for ideals of rings, using Zorn’s lemma. \hfill $\square$

In rings we have that if $I$ is a prime ideal then $I$ is irreducible.

Definition 2.10. An ideal $I$ of a semiring $R$ is irreducible if for $H, K \subset R$ ideals we have that

$$I = H \cap K \implies I = H \text{ or } I = K.$$  

An ideal $I$ of $R$ is strongly irreducible if for $H, K \subset R$ ideals we have that

$$I \supset H \cap K \implies I \supset H \text{ or } I \supset K.$$  

Proposition 2.11. Let $I \subset R$ be an ideal. Then $I$ is prime if and only if

1. $I$ is strongly irreducible, and
2. for any ideal $H \subset R$, $H^2 \subset I \implies H \subset I$.  

Proof. \((\Rightarrow)\) (2) is immediate. For (1), let \(M \cap K \subset I\). Then \(MK \subset M \cap K \subset I\) so \(M \subset I\) or \(K \subset I\).

\((\Leftarrow)\) Say \(M, K\) are ideals such that \(MK \subset I\). Then we have that \((M \cap K)^2 \subset MK \subset I\).

By (2) it follows that \(M \cap K \subset I\), so by (1) \(M \subset I\) or \(K \subset I\). Thus \(I\) is prime. \(\square\)

**Question 2.12.** (spoiler): Would this proposition be useful for finding the right definition of a prime tropical ideal?

**References**